Chapter 62

Partial Derivatives

FUNCTIONS OF SEVERAL VARIABLES. If a real number z is assigned to each point (x, y) of a part (region) of the xy plane, then z is said to be given as a function, $z = f(x, y)$, of the independent variables x and y. The locus of all points (x, y, z) satisfying $z = f(x, y)$ is a surface in ordinary space. In a similar manner, functions $w = f(x, y, z, \ldots)$ of several independent variables may be defined, but no geometric picture is available.

There are a number of differences between the calculus of one and of two variables. Fortunately, the calculus of functions of three or more variables differs only slightly from that of functions of two variables. The study here will be limited largely to functions of two variables.

LIMITS AND CONTINUITY. We say that a function $f(x, y)$ has a limit A as $x \to x_0$ and $y \to y_0$, and we write $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x, y) = A$, if, for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that, for all (x, y) satisfying

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \tag{62.1}$$

we have $|f(x, y) - A| < \epsilon$. Here, *(62.1)* defines a deleted neighborhood of (x_0, y_0), namely, all points except (x_0, y_0) lying within a circle of radius δ and center (x_0, y_0).

A function $f(x, y)$ is said to be continuous at (x_0, y_0) provided $f(x_0, y_0)$ is defined and $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x, y) = f(x_0, y_0)$. (See Problems 1 and 2.)

PARTIAL DERIVATIVES. Let $z = f(x, y)$ be a function of the independent variables x and y. Since x and y are independent, we may (1) allow x to vary while y is held fixed, (2) allow y to vary while x is held fixed, or (3) permit x and y to vary simultaneously. In the first two cases, z is in effect a function of a single variable and can be differentiated in accordance with the usual rules.

If x varies while y is held fixed, then z is a function of x; its derivative with respect to x,

$$f_x(x, y) = \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

is called *the (first) partial derivative of $z = f(x, y)$ with respect to x.*

If y varies while x is held fixed, z is a function of y; its derivative with respect to y,

$$f_y(x, y) = \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

is called *the (first) partial derivative of $z = f(x, y)$ with respect to y.* (See Problems 3 to 8.)

If z is defined implicitly as a function of x and y by the relation $F(x, y, z) = 0$, the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ may be found using the implicit differentiation rule of Chapter 11. (See Problems 9 to 12.)

The partial derivatives defined above have simple geometric interpretations. Consider the surface $z = f(x, y)$ in Fig. 62-1. Let APB and CPD be sections of the surface cut by planes through P, parallel to xOz and yOz, respectively. As x varies while y is held fixed, P moves along the curve APB and the value of $\partial z/\partial x$ at P is the slope of the curve APB at P.

Similarly, as y varies while x is held fixed, P moves along the curve CPD and the value of $\partial z/\partial y$ at P is the slope of the curve CPD at P. (See Problem 13.)

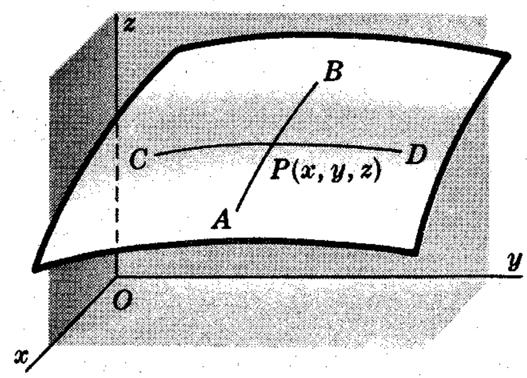

Fig. 62-1

PARTIAL DERIVATIVES OF HIGHER ORDERS. The partial derivative $\partial z/\partial x$ of $z = f(x, y)$ may in turn be differentiated partially with respect to x and y, yielding the second partial derivatives

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y\, \partial x} = f_{yx}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

Similarly, from $\partial z/\partial y$ we may obtain

$$\frac{\partial^2 z}{\partial x\, \partial y} = f_{xy}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

If $z = f(x, y)$ and its partial derivatives are continuous, the order of differentiation turns out to be immaterial; that is, $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial^2 z}{\partial y\, \partial x}$. (See Problems 14 and 15.)

Solved Problems

1. Investigate $z = x^2 + y^2$ for continuity.

For any set of finite values $(x, y) = (a, b)$, we have $z = a^2 + b^2$. As $x \to a$ and $y \to b$, $x^2 + y^2 \to a^2 + b^2$. Hence, the function is continuous everywhere.

2. The following functions are continuous everywhere except at the origin $(0, 0)$, where they are not defined. Can they be made continuous there?

(*a*) $z = \dfrac{\sin(x+y)}{x+y}$

Let $(x, y) \to (0, 0)$ over the line $y = mx$; then $z = \dfrac{\sin(x+y)}{x+y} = \dfrac{\sin(1+m)x}{(1+m)x} \to 1$. The function may be made continuous everywhere by redefining it as $z = \dfrac{\sin(x+y)}{x+y}$ for $(x, y) \neq (0, 0)$; $z = 1$ for $(x, y) = (0, 0)$.

(*b*) $z = \dfrac{xy}{x^2 + y^2}$

Let $(x, y) \to (0, 0)$ over the line $y = mx$; the limiting value of $z = \dfrac{xy}{x^2 + y^2} = \dfrac{m}{1 + m^2}$ depends on the particular line chosen. Thus, the function cannot be made continuous at $(0, 0)$.

In Problems 3 to 7, find the first partial derivatives.

3. $z = 2x^2 - 3xy + 4y^2$

Treating y as a constant and differentiating with respect to x yield $\dfrac{\partial z}{\partial x} = 4x - 3y$.

Treating x as a constant and differentiating with respect to y yield $\dfrac{\partial z}{\partial y} = -3x + 8y$.

4. $z = \dfrac{x^2}{y} + \dfrac{y^2}{x}$

Treating y as a constant and differentiating with respect to x yield $\dfrac{\partial z}{\partial x} = \dfrac{2x}{y} - \dfrac{y^2}{x^2}$.

Treating x as a constant and differentiating with respect to y yield $\dfrac{\partial z}{\partial y} = -\dfrac{x^2}{y^2} + \dfrac{2y}{x}$.

5. $z = \sin(2x + 3y)$

$$\frac{\partial z}{\partial x} = 2\cos(2x + 3y) \qquad \text{and} \qquad \frac{\partial z}{\partial y} = 3\cos(2x + 3y)$$

6. $z = \arctan x^2y + \arctan xy^2$

$$\frac{\partial z}{\partial x} = \frac{2xy}{1 + x^4y^2} + \frac{y^2}{1 + x^2y^4} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{x^2}{1 + x^4y^2} + \frac{2xy}{1 + x^2y^4}$$

7. $z = e^{x^2+xy}$

$$\frac{\partial z}{\partial x} = e^{x^2+xy}(2x + y) = z(2x + y) \qquad \text{and} \qquad \frac{\partial z}{\partial y} = e^{x^2+xy}(x) = xz$$

8. The area of a triangle is given by $K = \frac{1}{2}ab\sin C$. If $a = 20$, $b = 30$, and $C = 30°$, find:
(*a*) The rate of change of K with respect to a, when b and C are constant.
(*b*) The rate of change of K with respect to C, when a and b are constant.
(*c*) The rate of change of b with respect to a, when K and C are constant.

(*a*) $\dfrac{\partial K}{\partial a} = \dfrac{1}{2}\, b \sin C = \dfrac{1}{2}\,(30)(\sin 30°) = \dfrac{15}{2}$

(*b*) $\dfrac{\partial K}{\partial C} = \dfrac{1}{2}\, ab\cos C = \dfrac{1}{2}\,(20)(30)(\cos 30°) = 150\sqrt{3}$

(*c*) $b = \dfrac{2K}{a\sin C}$ and $\dfrac{\partial b}{\partial a} = -\dfrac{2K}{a^2 \sin C} = -\dfrac{2(\frac{1}{2}ab\sin C)}{a^2\sin C} = -\dfrac{b}{a} = -\dfrac{3}{2}$

In Problems 9 to 11, find the first partial derivatives of z with respect to the independent variables x and y.

9. $x^2 + y^2 + z^2 = 25$

Solution 1: Solve for z to obtain $z = \pm\sqrt{25 - x^2 - y^2}$. Then

$$\frac{\partial z}{\partial x} = \frac{-x}{\pm\sqrt{25 - x^2 - y^2}} = -\frac{x}{z} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\pm\sqrt{25 - x^2 - y^2}} = -\frac{y}{z}$$

Solution 2: Differentiate implicitly with respect to x, treating y as a constant, to obtain

$$2x + 2z\frac{\partial z}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial z}{\partial x} = -\frac{x}{z}$$

Then differentiate implicitly with respect to y, treating x as a constant:

$$2y + 2z\frac{\partial z}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

10. $x^2(2y+3z)+y^2(3x-4z)+z^2(x-2y)=xyz$

The procedure of Solution 1 of Problem 9 would be inconvenient here. Instead, differentiating implicitly with respect to x yields

$$2x(2y+3z)+3x^2\frac{\partial z}{\partial x}+3y^2-4y^2\frac{\partial z}{\partial x}+2z(x-2y)\frac{\partial z}{\partial x}+z^2=yz+xy\frac{\partial z}{\partial x}$$

so that
$$\frac{\partial z}{\partial x}=-\frac{4xy+6xz+3y^2+z^2-yz}{3x^2-4y^2+2xz-4yz-xy}$$

Differentiating implicitly with respect to y yields

$$2x^2+3x^2\frac{\partial z}{\partial y}+2y(3x-4z)-4y^2\frac{\partial z}{\partial y}+2z(x-2y)\frac{\partial z}{\partial y}-2z^2=xz+xy\frac{\partial z}{\partial y}$$

so that
$$\frac{\partial z}{\partial y}=-\frac{2x^2+6xy-8yz-2z^2-xz}{3x^2-4y^2+2xz-4yz-xy}$$

11. $xy+yz+zx=1$

Differentiating with respect to x yields $y+y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial x}+z=0$ and $\dfrac{\partial z}{\partial x}=-\dfrac{y+z}{x+y}$.

Differentiating with respect to y yields $x+y\dfrac{\partial z}{\partial y}+z+x\dfrac{\partial z}{\partial y}=0$ and $\dfrac{\partial z}{\partial y}=-\dfrac{x+z}{x+y}$.

12. Considering x and y as independent variables, find $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}, \dfrac{\partial \theta}{\partial x}, \dfrac{\partial \theta}{\partial y}$ when $x=e^{2r}\cos\theta$, $y=e^{3r}\sin\theta$.

First differentiate the given relations with respect to x:

$$1=2e^{2r}\cos\theta\frac{\partial r}{\partial x}-e^{2r}\sin\theta\frac{\partial \theta}{\partial x}\qquad\text{and}\qquad 0=3e^{3r}\sin\theta\frac{\partial r}{\partial x}+e^{3r}\cos\theta\frac{\partial \theta}{\partial x}$$

Then solve simultaneously to obtain $\dfrac{\partial r}{\partial x}=\dfrac{\cos\theta}{e^{2r}(2+\sin^2\theta)}$ and $\dfrac{\partial \theta}{\partial x}=-\dfrac{3\sin\theta}{e^{2r}(2+\sin^2\theta)}$.

Now differentiate the given relations with respect to y:

$$0=2e^{2r}\cos\theta\frac{\partial r}{\partial y}-e^{2r}\sin\theta\frac{\partial \theta}{\partial y}\qquad\text{and}\qquad 1=3e^{3r}\sin\theta\frac{\partial r}{\partial y}+e^{3r}\cos\theta\frac{\partial \theta}{\partial y}$$

Then solve simultaneously to obtain $\dfrac{\partial r}{\partial y}=\dfrac{\sin\theta}{e^{3r}(2+\sin^2\theta)}$ and $\dfrac{\partial \theta}{\partial y}=\dfrac{2\cos\theta}{e^{3r}(2+\sin^2\theta)}$.

13. Find the slopes of the curves cut from the surface $z=3x^2+4y^2-6$ by planes through the point $(1,1,1)$ and parallel to the coordinate planes xOz and yOz.

The plane $x=1$, parallel to the plane yOz, intersects the surface in the curve $z=4y^2-3$, $x=1$. Then $\partial z/\partial y=8y=8\times 1=8$ is the required slope.

The plane $y=1$, parallel to the plane xOz, intersects the surface in the curve $z=3x^2-2$, $y=1$. Then $\partial z/\partial x=6x=6$ is the required slope.

In Problems 14 and 15, find all second partial derivatives of z.

14. $z=x^2+3xy+y^2$

$$\frac{\partial z}{\partial x}=2x+3y\qquad \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=2\qquad \frac{\partial^2 z}{\partial y\,\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=3$$

$$\frac{\partial z}{\partial y}=3x+2y\qquad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=2\qquad \frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=3$$

15. $z = x\cos y - y\cos x$

$$\frac{\partial z}{\partial x} = \cos y + y\sin x \qquad \frac{\partial z}{\partial y} = -x\sin y - \cos x \qquad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = y\cos x$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -\sin y + \sin x = \frac{\partial^2 z}{\partial x\,\partial y} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = -x\cos y$$

Supplementary Problems

16. Investigate each of the following to determine whether or not it can be made continuous at $(0, 0)$: $(a)\ \dfrac{y^2}{x^2+y^2}$, $(b)\ \dfrac{x-y}{x+y}$, $(c)\ \dfrac{x^3+y^3}{x^2+y^2}$, $(d)\ \dfrac{x+y}{x^2+y^2}$. *Ans.* (a) no; (b) no; (c) yes; (d) no

17. For each of the following functions, find $\partial z/\partial x$ and $\partial z/\partial y$.

(a) $z = x^2 + 3xy + y^2$ *Ans.* $\dfrac{\partial z}{\partial x} = 2x + 3y;\ \dfrac{\partial z}{\partial y} = 3x + 2y$

(b) $z = \dfrac{x}{y^2} - \dfrac{y}{x^2}$ *Ans.* $\dfrac{\partial z}{\partial x} = \dfrac{1}{y^2} + \dfrac{2y}{x^3};\ \dfrac{\partial z}{\partial y} = -\dfrac{2x}{y^3} - \dfrac{1}{x^2}$

(c) $z = \sin 3x \cos 4y$ *Ans.* $\dfrac{\partial z}{\partial x} = 3\cos 3x\cos 4y;\ \dfrac{\partial z}{\partial y} = -4\sin 3x \sin 4y$

(d) $z = \arctan\dfrac{y}{x}$ *Ans.* $\dfrac{\partial z}{\partial x} = \dfrac{-y}{x^2+y^2};\ \dfrac{\partial z}{\partial y} = \dfrac{x}{x^2+y^2}$

(e) $x^2 - 4y^2 + 9z^2 = 36$ *Ans.* $\dfrac{\partial z}{\partial x} = -\dfrac{x}{9z};\ \dfrac{\partial z}{\partial y} = \dfrac{4y}{9z}$

(f) $z^3 - 3x^2y + 6xyz = 0$ *Ans.* $\dfrac{\partial z}{\partial x} = \dfrac{2y(x-z)}{z^2+2xy};\ \dfrac{\partial z}{\partial y} = \dfrac{x(x-2z)}{z^2+2xy}$

(g) $yz + xz + xy = 0$ *Ans.* $\dfrac{\partial z}{\partial x} = -\dfrac{y+z}{x+y};\ \dfrac{\partial z}{\partial y} = -\dfrac{x+z}{x+y}$

18. (a) If $z = \sqrt{x^2+y^2}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

(b) If $z = \ln\sqrt{x^2+y^2}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 1$.

(c) If $z = e^{x/y}\sin\dfrac{x}{y} + e^{y/x}\cos\dfrac{y}{x}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.

(d) If $z = (ax+by)^2 + e^{ax+by} + \sin(ax+by)$, show that $b\dfrac{\partial z}{\partial x} = a\dfrac{\partial z}{\partial y}$.

19. Find the equation of the line tangent to

(a) The parabola $z = 2x^2 - 3y^2$, $y = 1$ at the point $(-2, 1, 5)$ *Ans.* $8x + z + 11 = 0,\ y = 1$

(b) The parabola $z = 2x^2 - 3y^2$, $x = -2$ at the point $(-2, 1, 5)$ *Ans.* $6y + z - 11 = 0,\ x = -2$

(c) The hyperbola $z = 2x^2 - 3y^2$, $z = 5$ at the point $(-2, 1, 5)$ *Ans.* $4x + 3y + 5 = 0,\ z = 5$

Show that these three lines lie in the plane $8x + 6y + z + 5 = 0$.

20. For each of the following functions, find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x\,\partial y}$, $\dfrac{\partial^2 z}{\partial y\,\partial x}$, and $\dfrac{\partial^2 z}{\partial y^2}$.

(*a*) $z = 2x^2 - 5xy + y^2$ *Ans.* $\dfrac{\partial^2 z}{\partial x^2} = 4;\ \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x} = -5;\ \dfrac{\partial^2 z}{\partial y^2} = 2$

(*b*) $z = \dfrac{x}{y^2} - \dfrac{y}{x^2}$ *Ans.* $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{6y}{x^4};\ \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x} = 2\left(\dfrac{1}{x^3} - \dfrac{1}{y^3}\right);\ \dfrac{\partial^2 z}{\partial y^2} = \dfrac{6x}{y^4}$

(*c*) $z = \sin 3x \cos 4y$ *Ans.* $\dfrac{\partial^2 z}{\partial x^2} = -9z;\ \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x} = -12\cos 3x \sin 4y;\ \dfrac{\partial^2 z}{\partial y^2} = -16z$

(*d*) $z = \arctan \dfrac{y}{x}$ *Ans.* $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{\partial^2 z}{\partial y^2} = \dfrac{2xy}{(x^2+y^2)^2};\ \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$

21. (*a*) If $z = \dfrac{xy}{x-y}$, show that $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} = 0$.

(*b*) If $z = e^{\alpha x} \cos \beta y$ and $\beta = \pm\alpha$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

(*c*) If $z = e^{-t}(\sin x + \cos y)$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial z}{\partial t}$.

(*d*) If $z = \sin ax \sin by \sin kt\sqrt{a^2 + b^2}$, show that $\dfrac{\partial^2 z}{\partial t^2} = k^2\left(\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}\right)$.

22. For the gas formula $\left(p + \dfrac{a}{v^2}\right)(v - b) = ct$, where a, b, and c are constants, show that

$$\frac{\partial p}{\partial v} = \frac{2a(v-b) - (p + a/v^2)v^3}{v^3(v-b)} \qquad \frac{\partial v}{\partial t} = \frac{cv^3}{(p + a/v^2)v^3 - 2a(v-b)}$$

$$\frac{\partial t}{\partial p} = \frac{v-b}{c} \qquad \frac{\partial p}{\partial v}\,\frac{\partial v}{\partial t}\,\frac{\partial t}{\partial p} = -1$$

Chapter 63

Total Differentials and Total Derivatives

TOTAL DIFFERENTIALS. The differentials dx and dy for the function $y = f(x)$ of a single independent variable x were defined in Chapter 28 as

$$dx = \Delta x \qquad \text{and} \qquad dy = f'(x)\,dx = \frac{dy}{dx}\,dx$$

Consider the function $z = f(x, y)$ of the two independent variables x and y, and define $dx = \Delta x$ and $dy = \Delta y$. When x varies while y is held fixed, z is a function of x only and *the partial differential of z with respect to x* is defined as $d_x z = f_x(x, y)\,dx = \frac{\partial z}{\partial x}\,dx$. Similarly, *the partial differential of z with respect to y* is defined as $d_y z = f_y(x, y)\,dy = \frac{\partial z}{\partial y}\,dy$. The *total differential dz* is defined as the sum of the partial differentials,

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \tag{63.1}$$

For a function $w = F(x, y, z, \ldots, t)$, the total differential dw is defined as

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz + \cdots + \frac{\partial w}{\partial t}\,dt \tag{63.2}$$

(See Problems 1 and 2.)

As in the case of a function of a single variable, the total differential of a function of several variables gives a good approximation of the total increment of the function when the increments of the several independent variables are small.

EXAMPLE 1: When $z = xy$, $dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = y\,dx + x\,dy$; and when x and y are given increments $\Delta x = dx$ and $\Delta y = dy$, the increment Δz taken on by z is

$$\begin{aligned}\Delta z &= (x + \Delta x)(y + \Delta y) - xy = x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y\\ &= x\,dy + y\,dx + dx\,dy\end{aligned}$$

A geometric interpretation is given in Fig. 63-1: dz and Δz differ by the rectangle of area $\Delta x\,\Delta y = dx\,dy$.

(See Problems 3 to 9.)

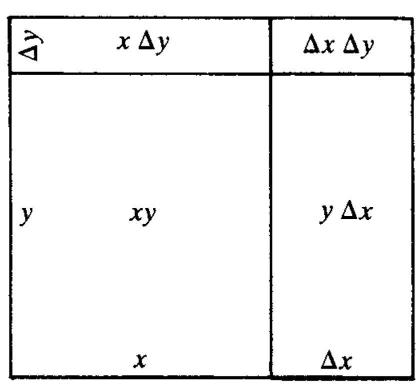

Fig. 63-1

THE CHAIN RULE FOR COMPOSITE FUNCTIONS. If $z = f(x, y)$ is a continuous function of the variables x and y with continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, and if x and y are

differentiable functions $x = g(t)$ and $y = h(t)$ of a variable t, then z is a function of t and dz/dt, called the *total derivative* of z with respect to t, is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{63.3}$$

Similarly, if $w = f(x, y, z, \ldots)$ is a continuous function of the variables $x, y, z, \ldots$ with continuous partial derivatives, and if $x, y, z, \ldots$ are differentiable functions of a variable t, the total derivative of w with respect to t is given by

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \cdots \tag{63.4}$$

(See Problems 10 to 16.)

If $z = f(x, y)$ is a continuous function of the variables x and y with continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, and if x and y are continuous functions $x = g(r, s)$ and $y = h(r, s)$ of the independent variables r and s, then z is a function of r and s with

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \tag{63.5}$$

Similarly, if $w = f(x, y, z, \ldots)$ is a continuous function of the variables $x, y, z, \ldots$ with continuous partial derivatives $\partial w/\partial x$, $\partial w/\partial y$, $\partial w/\partial z, \ldots$, and if $x, y, z, \ldots$ are continuous functions of the independent variables $r, s, t, \ldots$, then

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} + \cdots \\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} + \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{63.6}$$

(See Problems 17 to 19.)

Solved Problems

In Problems 1 and 2, find the total differential.

1. $z = x^3y + x^2y^2 + xy^3$

We have $\quad \dfrac{\partial z}{\partial x} = 3x^2y + 2xy^2 + y^3 \quad$ and $\quad \dfrac{\partial z}{\partial y} = x^3 + 2x^2y + 3xy^2$

Then $\quad dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy = (3x^2y + 2xy^2 + y^3)\,dx + (x^3 + 2x^2y + 3xy^2)\,dy$

2. $z = x \sin y - y \sin x$

We have $\quad \dfrac{\partial z}{\partial x} = \sin y - y \cos x \quad$ and $\quad \dfrac{\partial z}{\partial y} = x \cos y - \sin x$

Then $\quad dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy = (\sin y - y \cos x)\,dx + (x \cos y - \sin x)\,dy$

3. Compare dz and Δz, given $z = x^2 + 2xy - 3y^2$.

$$\frac{\partial z}{\partial x} = 2x + 2y \quad \text{and} \quad \frac{\partial z}{\partial y} = 2x - 6y. \quad \text{So} \quad dz = 2(x + y)\,dx + 2(x - 3y)\,dy$$

Also,
$$\begin{aligned}\Delta z &= [(x + dx)^2 + 2(x + dx)(y + dy) - 3(y + dy)^2] - (x^2 + 2xy - 3y^2)\\ &= 2(x + y)\,dx + 2(x - 3y)\,dy + (dx)^2 + 2\,dx\,dy - 3(dy)^2\end{aligned}$$

Thus dz and Δz differ by $(dx)^2 + 2\,dx\,dy - 3(dy)^2$.

4. Approximate the area of a rectangle of dimensions 35.02 by 24.97 units.

For dimensions x by y, the area is $A = xy$ so that $dA = \frac{\partial A}{\partial x}\,dx + \frac{\partial A}{\partial y}\,dy = y\,dx + x\,dy$. With $x = 35$, $dx = 0.02$, $y = 25$, and $dy = -0.03$, we have $A = 35(25) = 875$ and $dA = 25(0.02) + 35(-0.03) = -0.55$. The area is approximately $A + dA = 874.45$ square units.

5. Approximate the change in the hypotenuse of a right triangle of legs 6 and 8 inches when the shorter leg is lengthened by $\frac{1}{4}$ inch and the longer leg is shortened by $\frac{1}{8}$ inch.

Let x, y, and z be the shorter leg, the longer leg, and the hypotenuse of the triangle. Then

$$z = \sqrt{x^2 + y^2} \qquad \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \qquad dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$$

When $x = 6$, $y = 8$, $dx = \frac{1}{4}$, and $dy = -\frac{1}{8}$, then $dz = \dfrac{6(\frac{1}{4}) + 8(-\frac{1}{8})}{\sqrt{6^2 + 8^2}} = \dfrac{1}{20}$ inch. Thus the hypotenuse is lengthened by approximately $\frac{1}{20}$ inch.

6. The power consumed in an electrical resistor is given by $P = E^2/R$ (in watts). If $E = 200$ volts and $R = 8$ ohms, by how much does the power change if E is decreased by 5 volts and R is decreased by 0.2 ohm?

We have
$$\frac{\partial P}{\partial E} = \frac{2E}{R} \qquad \frac{\partial P}{\partial R} = -\frac{E^2}{R^2} \qquad dP = \frac{2E}{R}\,dE - \frac{E^2}{R^2}\,dR$$

When $E = 200$, $R = 8$, $dE = -5$, and $dR = -0.2$, then

$$dP = \frac{2(200)}{8}(-5) - \left(\frac{200}{8}\right)^2(-0.2) = -250 + 125 = -125$$

The power is reduced by approximately 125 watts.

7. The dimensions of a rectangular block of wood were found to be 10, 12, and 20 inches, with a possible error of 0.05 in in each of the measurements. Find (approximately) the greatest error in the surface area of the block and the percentage error in the area caused by the errors in the individual measurements.

The surface area is $S = 2(xy + yz + zx)$; then

$$dS = \frac{\partial S}{\partial x}\,dx + \frac{\partial S}{\partial y}\,dy + \frac{\partial S}{dz}\,dz = 2(y + z)\,dx + 2(x + z)\,dy + 2(y + x)\,dz$$

The greatest error in S occurs when the errors in the lengths are of the same sign, say positive. Then

$$dS = 2(12 + 20)(0.05) + 2(10 + 20)(0.05) + 2(12 + 10)(0.05) = 8.4\ \text{in}^2$$

The percentage error is (error/area)(100) = (8.4/1120)(100) = 0.75%.

8. For the formula $R = E/C$, find the maximum error and the percentage error if $C = 20$ with a possible error of 0.1 and $E = 120$ with a possible error of 0.05.

Here $$dR = \frac{\partial R}{\partial E}\, dE + \frac{\partial R}{\partial C}\, dC = \frac{1}{C}\, dE - \frac{E}{C^2}\, dC$$

The maximum error will occur when $dE = 0.05$ and $dC = -0.1$; then $dR = \frac{0.05}{20} - \frac{120}{400}(-0.1) = 0.0325$ is the approximate maximum error. The percentage error is $\frac{dR}{R}(100) = \frac{0.0325}{8}(100) = 0.40625 = 0.41\%$.

9. Two sides of a triangle were measured as 150 and 200 ft, and the included angle as 60°. If the possible errors are 0.2 ft in measuring the sides and 1° in the angle, what is the greatest possible error in the computed area?

Here $$A = \frac{1}{2} xy \sin\theta \qquad \frac{\partial A}{\partial x} = \frac{1}{2} y \sin\theta \qquad \frac{\partial A}{\partial y} = \frac{1}{2} x \sin\theta \qquad \frac{\partial A}{\partial \theta} = \frac{1}{2} xy \cos\theta$$

and $$dA = \frac{1}{2} y \sin\theta\, dx + \frac{1}{2} x \sin\theta\, dy + \frac{1}{2} xy \cos\theta\, d\theta$$

When $x = 150$, $y = 200$, $\theta = 60°$, $dx = 0.2$, $dy = 0.2$, and $d\theta = 1° = \pi/180$, then

$$dA = \tfrac{1}{2}(200)(\sin 60°)(0.2) + \tfrac{1}{2}(150)(\sin 60°)(0.2) + \tfrac{1}{2}(150)(200)(\cos 60°)(\pi/180) = 161.21 \text{ ft}^2$$

10. Find dz/dt, given $z = x^2 + 3xy + 5y^2$; $x = \sin t$, $y = \cos t$.

Since $$\frac{\partial z}{\partial x} = 2x + 3y \qquad \frac{\partial z}{\partial y} = 3x + 10y \qquad \frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\sin t$$

we have $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x + 3y)\cos t - (3x + 10y)\sin t$$

11. Find dz/dt, given $z = \ln(x^2 + y^2)$; $x = e^{-t}$, $y = e^t$.

Since $$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2} \qquad \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2} \qquad \frac{dx}{dt} = -e^{-t} \qquad \frac{dy}{dt} = e^t$$

we have $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{2x}{x^2 + y^2}(-e^{-t}) + \frac{2y}{x^2 + y^2} e^t = 2\,\frac{ye^t - xe^{-t}}{x^2 + y^2}$$

12. Let $z = f(x, y)$ be a continuous function of x and y with continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, and let y be a differentiable function of x. Then z is a differentiable function of x. Find a formula for dz/dx.

By *(63.3)*, $$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

The shift in notation from z to f is made here to avoid possible confusion arising from the use of dz/dx and $\partial z/\partial x$ in the same expression.

13. Find dz/dx, given $z = f(x, y) = x^2 + 2xy + 4y^2$, $y = e^{ax}$.

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = (2x + 2y) + (2x + 8y)ae^{ax} = 2(x + y) + 2a(x + 4y)e^{ax}$$

14. Find (*a*) dz/dx and (*b*) dz/dy, given $z = f(x, y) = xy^2 + x^2y$, $y = \ln x$.

(*a*) Here x is the independent variable:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = (y^2 + 2xy) + (2xy + x^2)\frac{1}{x} = y^2 + 2xy + 2y + x$$

(*b*) Here y is the independent variable:

$$\frac{dz}{dy} = \frac{\partial f}{\partial x}\frac{dx}{dy} + \frac{\partial f}{\partial y} = (y^2 + 2xy)x + (2xy + x^2) = xy^2 + 2x^2y + 2xy + x^2$$

15. The altitude of a right circular cone is 15 inches and is increasing at 0.2 in/min. The radius of the base is 10 inches and is decreasing at 0.3 in/min. How fast is the volume changing?

Let x be the radius, and y the altitude of the cone (Fig. 63-2). From $V = \frac{1}{3}\pi x^2 y$, considering x and y as functions of time t, we have

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \frac{\pi}{3}\left(2xy\frac{dx}{dt} + x^2\frac{dy}{dt}\right) = \frac{\pi}{3}[2(10)(15)(-0.3) + 10^2(0.2)] = -70\pi/3 \text{ in}^3/\text{min}$$

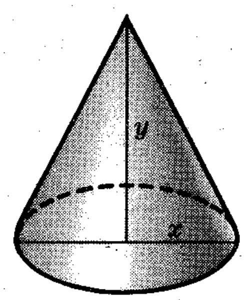

Fig. 63-2

16. A point P is moving along the curve of intersection of the paraboloid $\frac{x^2}{16} - \frac{y^2}{9} = z$ and the cylinder $x^2 + y^2 = 5$, with x, y, and z expressed in inches. If x is increasing at 0.2 in/min, how fast is z changing when $x = 2$?

From $z = \frac{x^2}{16} - \frac{y^2}{9}$, we obtain $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{x}{8}\frac{dx}{dt} - \frac{2y}{9}\frac{dy}{dt}$. Since $x^2 + y^2 = 5$, $y = \pm 1$ when $x = 2$; also, differentiation yields $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$.

When $y = 1$, $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{2}{1}(0.2) = -0.4$ and $\frac{dz}{dt} = \frac{2}{8}(0.2) - \frac{2}{9}(-0.4) = \frac{5}{36}$ in/min.

When $y = -1$, $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = 0.4$ and $\frac{dz}{dt} = \frac{2}{8}(0.2) - \frac{2}{9}(-1)(0.4) = \frac{5}{36}$ in/min.

17. Find $\partial z/\partial r$ and $\partial z/\partial s$, given $z = x^2 + xy + y^2$; $x = 2r + s$, $y = r - 2s$.

Here $\frac{\partial z}{\partial x} = 2x + y \qquad \frac{\partial z}{\partial y} = x + 2y \qquad \frac{\partial x}{\partial r} = 2 \qquad \frac{\partial x}{\partial s} = 1 \qquad \frac{\partial y}{\partial r} = 1 \qquad \frac{\partial y}{\partial s} = -2$

Then
$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (2x + y)(2) + (x + 2y)(1) = 5x + 4y$$

and
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x + y)(1) + (x + 2y)(-2) = -3y$$

18. Find $\frac{\partial u}{\partial \rho}$, $\frac{\partial u}{\partial \beta}$, and $\frac{\partial u}{\partial \theta}$, given $u = x^2 + 2y^2 + 2z^2$; $x = \rho \sin\beta\cos\theta$, $y = \rho\sin\beta\sin\theta$, $z = \rho\cos\beta$.

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \rho} = 2x\sin\beta\cos\theta + 4y\sin\beta\sin\theta + 4z\cos\beta$$

$$\frac{\partial u}{\partial \beta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \beta}=2x\,\rho\cos\beta\cos\theta+4y\,\rho\cos\beta\sin\theta-4z\,\rho\sin\beta$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta}=-2x\,\rho\sin\beta\sin\theta+4y\,\rho\sin\beta\cos\theta$$

19. Find du/dx, given $u=f(x,y,z)=xy+yz+zx$; $y=1/x$, $z=x^2$.

From *(63.6)*,

$$\frac{du}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial z}\frac{dz}{dx}=(y+z)+(x+z)\left(-\frac{1}{x^2}\right)+(y+x)2x=y+z+2x(x+y)-\frac{x+z}{x^2}$$

20. If $z=f(x,y)$ is a continuous function of x and y possessing continuous first partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, derive the basic formula

$$\Delta z=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y+\epsilon_1\,\Delta x+\epsilon_2\,\Delta y \tag{1}$$

where ϵ_1 and $\epsilon_2\to 0$ as Δx and $\Delta y\to 0$.

When x and y are given increments Δx and Δy respectively, the increment given to z is

$$\begin{aligned}\Delta z&=f(x+\Delta x,y+\Delta y)-f(x,y)\\&=[f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)]+[f(x,y+\Delta y)-f(x,y)]\end{aligned} \tag{2}$$

In the first bracketed expression, only x changes; in the second, only y changes. Thus, the law of the mean *(26.5)* may be applied to each:

$$f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)=\Delta x\,f_x(x+\theta_1\,\Delta x,y+\Delta y) \tag{3}$$

$$f(x,y+\Delta y)-f(x,y)=\Delta y\,f_y(x,y+\theta_2\,\Delta y) \tag{4}$$

where $0<\theta_1<1$ and $0<\theta_2<1$. Note that here the derivatives involved are partial derivatives.

Since $\partial z/\partial x=f_x(x,y)$ and $\partial z/\partial y=f_y(x,y)$ are, by hypothesis, continuous functions of x and y,

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f_x(x+\theta_1\,\Delta x,y+\Delta y)=f_x(x,y) \quad\text{and}\quad \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f_y(x,y+\theta_2\,\Delta y)=f_y(x,y)$$

Then $\quad f_x(x+\theta_1\,\Delta x,y+\Delta y)=f_x(x,y)+\epsilon_1 \quad$ and $\quad f_y(x,y+\theta_2\,\Delta y)=f_y(x,y)+\epsilon_2$

where $\epsilon_1\to 0$ and $\epsilon_2\to 0$ as Δx and $\Delta y\to 0$.

After making these replacements in *(3)* and *(4)* and then substituting in *(1)*, we have, as required,

$$\Delta z=[f_x(x,y)+\epsilon_1]\,\Delta x+[f_y(x,y)+\epsilon_2]\,\Delta y=f_x(x,y)\,\Delta x+f_y(x,y)\,\Delta y+\epsilon_1\,\Delta x+\epsilon_2\,\Delta y$$

Note that the total derivative dz is a fairly good approximation of the total increment Δz when $|\Delta x|$ and $|\Delta y|$ are small.

Supplementary Problems

21. Find the total differential, given:

(*a*) $z=x^3y+2xy^3$ *Ans.* $dz=(3x^2+2y^2)y\,dx+(x^2+6y^2)x\,dy$

(*b*) $\theta=\arctan(y/x)$ *Ans.* $d\theta=\dfrac{x\,dy-y\,dx}{x^2+y^2}$

(*c*) $z=e^{x^2-y^2}$ *Ans.* $dz=2z(x\,dx-y\,dy)$

(*d*) $z=x(x^2+y^2)^{-1/2}$ *Ans.* $dz=\dfrac{y(y\,dx-x\,dy)}{(x^2+y^2)^{3/2}}$

22. The fundamental frequency of vibration of a string or wire of circular section under tension T is $n = \frac{1}{2rl}\sqrt{\frac{T}{\pi d}}$, where l is the length, r the radius, and d the density of the string. Find (*a*) the approximate effect of changing l by a small amount dl, (*b*) the effect of changing T by a small amount dT, and (*c*) the effect of changing l and T simultaneously.

Ans. (*a*) $-(n/l)\,dl$; (*b*) $(n/2T)\,dT$; (*c*) $n(-dl/l + dT/2T)$

23. Use differentials to compute (*a*) the volume of a box with square base of side 8.005 and height 9.996 ft; (*b*) the diagonal of a rectangular box of dimensions 3.03 by 5.98 by 6.01 ft.

Ans. (*a*) 640.544 ft^3; (*b*) 9.003 ft

24. Approximate the maximum possible error and the percentage of error when z is computed by the given formula:
(*a*) $z = \pi r^2 h$; $r = 5 \pm 0.05$, $h = 12 \pm 0.1$ *Ans.* 8.5π; 2.8%
(*b*) $1/z = 1/f + 1/g$; $f = 4 \pm 0.01$, $g = 8 \pm 0.02$ *Ans.* 0.0067; 0.25%
(*c*) $z = y/x$; $x = 1.8 \pm 0.1$, $y = 2.4 \pm 0.1$ *Ans.* 0.13; 10%

25. Find the approximate maximum percentage of error in:
(*a*) $\omega = \sqrt[3]{g/b}$ if there is a possible 1% error in measuring g and a possible $\frac{1}{2}$% error in measuring b.
$\left(\textit{Hint}: \ln \omega = \frac{1}{3}(\ln g - \ln b);\ \frac{\partial\omega}{\omega} = \frac{1}{3}\left(\frac{dg}{g} - \frac{db}{b}\right);\ \left|\frac{dg}{g}\right| = 0.01;\ \left|\frac{db}{b}\right| = 0.005\right)$ *Ans.* 0.005
(*b*) $g = 2s/t^2$ if there is a possible 1% error in measuring s and $\frac{1}{4}$% error in measuring t.
Ans. 0.015

26. Find du/dt, given:
(*a*) $u = x^2y^3$; $x = 2t^3$, $y = 3t^2$ *Ans.* $6xy^2t(2yt + 3x)$
(*b*) $u = x\cos y + y\sin x$; $x = \sin 2t$, $y = \cos 2t$
Ans. $2(\cos y + y\cos x)\cos 2t - 2(-x\sin y + \sin x)\sin 2t$
(*c*) $u = xy + yz + zx$; $x = e^t$, $y = e^{-t}$, $z = e^t + e^{-t}$ *Ans.* $(x + 2y + z)e^t - (2x + y + z)e^{-t}$

27. At a certain instant the radius of a right circular cylinder is 6 inches and is increasing at the rate 0.2 in/sec, while the altitude is 8 inches and is decreasing at the rate 0.4 in/s. Find the time rate of change (*a*) of the volume and (*b*) of the surface at that instant.

Ans. (*a*) 4.8π in^3/sec; (*b*) 3.2π in^2/sec

28. A particle moves in a plane so that at any time t its abscissa and ordinate are given by $x = 2 + 3t$, $y = t^2 + 4$ with x and y in feet and t in minutes. How is the distance of the particle from the origin changing when $t = 1$? *Ans.* $5/\sqrt{2}$ ft/min

29. A point is moving along the curve of intersection of $x^2 + 3xy + 3y^2 = z^2$ and the plane $x - 2y + 4 = 0$. When $x = 2$ and is increasing at 3 units/sec, find (*a*) how y is changing, (*b*) how z is changing, and (*c*) the speed of the point.

Ans. (*a*) increasing 3/2 units/sec; (*b*) increasing 75/14 units/sec at $(2, 3, 7)$ and decreasing 75/14 units/sec at $(2, 3, -7)$; (*c*) 6.3 units/sec

30. Find $\partial z/\partial s$ and $\partial z/\partial t$, given:
(*a*) $z = x^2 - 2y^2$; $x = 3s + 2t$, $y = 3s - 2t$ *Ans.* $6(x - 2y)$; $4(x + 2y)$
(*b*) $z = x^2 + 3xy + y^2$; $x = \sin s + \cos t$, $y = \sin s - \cos t$ *Ans.* $5(x + y)\cos s$; $(x - y)\sin t$
(*c*) $z = x^2 + 2y^2$; $x = e^s - e^t$, $y = e^s + e^t$ *Ans.* $2(x + 2y)e^s$; $2(2y - x)e^t$
(*d*) $z = \sin(4x + 5y)$; $x = s + t$, $y = s - t$ *Ans.* $9\cos(4x + 5y)$; $-\cos(4x + 5y)$
(*e*) $z = e^{xy}$; $x = s^2 + 2st$, $y = 2st + t^2$ *Ans.* $2e^{xy}[tx + (s + t)y]$; $2e^{xy}[(s + t)x + sy]$

31. (*a*) If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(*b*) If $u = f(x, y)$ and $x = r\cosh s$, $y = r\sinh s$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 - \frac{1}{s^2}\left(\frac{\partial u}{\partial s}\right)^2$$

32. (*a*) If $z = f(x + \alpha y) + g(x - \alpha y)$, show that $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial^2 z}{\partial y^2}$. (*Hint*: Write $z = f(u) + g(v)$, $u = x + \alpha y$, $v = x - \alpha y$.)

(*b*) If $z = x^n f(y/x)$, show that $x\,\partial z/\partial x + y\,\partial z/\partial y = nz$.

(*c*) If $z = f(x, y)$ and $x = g(t)$, $y = h(t)$, show that, subject to continuity conditions

$$\frac{d^2 z}{dt^2} = f_{xx}(g')^2 + 2f_{xy}g'h' + f_{yy}(h')^2 + f_x g'' + f_y h''$$

(*d*) If $z = f(x, y)$; $x = g(r, s)$, $y = h(r, s)$, show that, subject to continuity conditions

$$\frac{\partial^2 z}{\partial r^2} = f_{xx}(g_r)^2 + 2f_{xy}g_r h_r + f_{yy}(h_r)^2 + f_x g_{rr} + f_y h_{rr}$$

$$\frac{\partial^2 z}{\partial r\,\partial s} = f_{xx}g_r g_s + f_{xy}(g_r h_s + g_s h_r) + f_{yy}h_r h_s + f_x g_{rs} + f_y h_{rs}$$

$$\frac{\partial^2 z}{\partial s^2} = f_{xx}(g_s)^2 + 2f_{xy}g_s h_s + f_{yy}(h_s)^2 + f_x g_{ss} + f_y h_{ss}$$

33. A function $f(x, y)$ is called *homogeneous of order n* if $f(tx, ty) = t^n f(x, y)$. (For example, $f(x, y) = x^2 + 2xy + 3y^2$ is homogeneous of order 2; $f(x, y) = x\sin(y/x) + y\cos(y/x)$ is homogeneous of order 1.) Differentiate $f(tx, ty) = t^n f(x, y)$ with respect to t and replace t by 1 to show that $xf_x + yf_y = nf$. Verify this formula using the two given examples. See also Problem 32(*b*).

34. If $z = \phi(u, v)$, where $u = f(x, y)$ and $v = g(x, y)$, and if $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, show that

(*a*) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ (*b*) $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = \left\{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right\}\left(\dfrac{\partial^2 \phi}{\partial u^2} + \dfrac{\partial^2 \phi}{\partial v^2}\right)$

35. Use (*1*) of Problem 20 to derive the chain rules (*63.3*) and (*63.5*). (*Hint*: For (*63.3*), divide by Δt.)

Chapter 64

Implicit Functions

THE DIFFERENTIATION of a function of one variable, defined implicitly by a relation $f(x, y) = 0$, was treated intuitively in Chapter 11. For this case, we state without proof:

Theorem 64.1: If $f(x, y)$ is continuous in a region including a point (x_0, y_0) for which $f(x_0, y_0) = 0$, if $\partial f/\partial x$ and $\partial f/\partial y$ are continuous throughout the region, and if $\partial f/\partial y \neq 0$ at (x_0, y_0), then there is a neighborhood of (x_0, y_0) in which $f(x, y) = 0$ can be solved for y as a continuous differentiable function of x, $y = \phi(x)$, with $y_0 = \phi(x_0)$ and $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$.

(See Problems 1 to 3.)

Extending this theorem, we have the following:

Theorem 64.2: If $F(x, y, z)$ is continuous in a region including a point (x_0, y_0, z_0) for which $F(x_0, y_0, z_0) = 0$, if $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$, and $\dfrac{\partial F}{\partial z}$ are continuous throughout the region, and if $\partial F/\partial z \neq 0$ at (x_0, y_0, z_0), then there is a neighborhood of (x_0, y_0, z_0) in which $F(x, y, z) = 0$ can be solved for z as a continuous differentiable function of x and y, $z = \phi(x, y)$, with $z_0 = \phi(x_0, y_0)$ and $\dfrac{\partial z}{\partial x} = -\dfrac{\partial F/\partial x}{\partial F/\partial z}$, $\dfrac{\partial z}{\partial y} = -\dfrac{\partial F/\partial y}{\partial F/\partial z}$.

(See Problems 4 and 5.)

Theorem 64.3: If $f(x, y, u, v)$ and $g(x, y, u, v)$ are continuous in a region including the point (x_0, y_0, u_0, v_0) for which $f(x_0, y_0, u_0, v_0) = 0$ and $g(x_0, y_0, u_0, v_0) = 0$, if the first partial derivatives of f and of g are continuous throughout the region, and if at (x_0, y_0, u_0, v_0) the determinant $J\left(\dfrac{f, g}{u, v}\right) \equiv \begin{vmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial g/\partial u & \partial g/\partial v \end{vmatrix} \neq 0$, then there is a neighborhood of (x_0, y_0, u_0, v_0) in which $f(x, y, u, v) = 0$ and $g(x, y, u, v) = 0$ can be solved simultaneously for u and v as continuous differentiable functions of x and y, $u = \phi(x, y)$ and $v = \psi(x, y)$. If at (x_0, y_0, u_0, v_0) the determinant $J\left(\dfrac{f, g}{x, y}\right) \neq 0$, then there is a neighborhood of (x_0, y_0, u_0, v_0) in which $f(x, y, u, v) = 0$ and $g(x, y, u, v) = 0$ can be solved for x and y as continuous differentiable functions of u and v, $x = h(u, v)$ and $y = k(u, v)$.

(See Problems 6 and 7.)

Solved Problems

1. Use Theorem 64.1 to show that $x^2 + y^2 - 13 = 0$ defines y as a continuous differentiable function of x in any neighborhood of the point $(2, 3)$ that does not include a point of the x axis. Find the derivative at the point.

Set $f(x, y) = x^2 + y^2 - 13$. Then $f(2, 3) = 0$, while in any neighborhood of $(2, 3)$ in which the function is defined, its partial derivatives $\partial f/\partial x = 2x$ and $\partial f/\partial y = 2y$ are continuous, and $\partial f/\partial y \neq 0$. Then

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \qquad \text{and} \qquad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{x}{y} = -\frac{2}{3} \text{ at } (2, 3)$$

2. Find dy/dx, given $f(x, y) = y^3 + xy - 12 = 0$.

We have $\dfrac{\partial f}{\partial x} = y$ and $\dfrac{\partial f}{\partial y} = 3y^2 + x$. So $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{y}{3y^2 + x}$

3. Find dy/dx, given $e^x \sin y + e^y \sin x = 1$.

Put $f(x, y) = e^x \sin y + e^y \sin x - 1$. Then $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{e^x \sin y + e^y \cos x}{e^x \cos y + e^y \sin x}$.

4. Find $\partial z/\partial x$ and $\partial z/\partial y$, given $F(x, y, z) = x^2 + 3xy - 2y^2 + 3xz + z^2 = 0$.

Treating z as a function of x and y defined by the relation and differentiating partially with respect to x and again with respect to y, we have

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = (2x + 3y + 3z) + (3x + 2z)\frac{\partial z}{\partial x} = 0 \tag{1}$$

and

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = (3x - 4y) + (3x + 2z)\frac{\partial z}{\partial y} = 0 \tag{2}$$

From (1), $\dfrac{\partial z}{\partial x} = -\dfrac{\partial F/\partial x}{\partial F/\partial z} = -\dfrac{2x + 3y + 3z}{3x + 2z}$. From (2), $\dfrac{\partial z}{\partial y} = -\dfrac{\partial F/\partial y}{\partial F/\partial z} = -\dfrac{3x - 4y}{3x + 2z}$.

5. Find $\partial z/\partial x$ and $\partial z/\partial y$, given $\sin xy + \sin yz + \sin zx = 1$.

Set $F(x, y, z) = \sin xy + \sin yz + \sin zx - 1$; then

$$\frac{\partial F}{\partial x} = y \cos xy + z \cos zx \qquad \frac{\partial F}{\partial y} = x \cos xy + z \cos yz \qquad \frac{\partial F}{\partial z} = y \cos yz + x \cos zx$$

and

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{y \cos xy + z \cos zx}{y \cos yz + x \cos zx} \qquad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{x \cos xy + z \cos yz}{y \cos yz + x \cos zx}$$

6. If u and v are defined as functions of x and y by the equations

$$f(x, y, u, v) = x + y^2 + 2uv = 0 \qquad g(x, y, u, v) = x^2 - xy + y^2 + u^2 + v^2 = 0$$

find (a) $\partial u/\partial x$, $\partial v/\partial x$ and (b) $\partial u/\partial y$, $\partial v/\partial y$.

(a) Differentiating f and g partially with respect to x, we obtain

$$1 + 2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x} = 0 \qquad \text{and} \qquad 2x - y + 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

Solving these relations simultaneously for $\partial u/\partial x$ and $\partial v/\partial x$, we find

$$\frac{\partial u}{\partial x} = \frac{v + u(y - 2x)}{2(u^2 - v^2)} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = \frac{v(2x - y) - u}{2(u^2 - v^2)}$$

(b) Differentiating f and g partially with respect to y, we obtain

$$2y + 2v\frac{\partial u}{\partial y} + 2u\frac{\partial v}{\partial y} = 0 \qquad \text{and} \qquad -x + 2y + 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

Then

$$\frac{\partial u}{\partial y} = \frac{u(x - 2y) + 2vy}{2(u^2 - v^2)} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{v(2y - x) - 2uy}{2(u^2 - v^2)}$$

7. Given $u^2 - v^2 + 2x + 3y = 0$ and $uv + x - y = 0$, find (a) $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$ and (b) $\dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}$.

(a) Here x and y are to be considered as independent variables. Differentiate the given equations partially with respect to x, obtaining

$$2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} + 2 = 0 \qquad \text{and} \qquad v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x} + 1 = 0$$

Solve these relations simultaneously to obtain $\frac{\partial u}{\partial x} = -\frac{u+v}{u^2+v^2}$ and $\frac{\partial v}{\partial x} = \frac{v-u}{u^2+v^2}$.

Differentiate the given equations partially with respect to y, obtaining

$$2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} + 3 = 0 \qquad \text{and} \qquad v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y} - 1 = 0$$

Solve simultaneously to obtain $\frac{\partial u}{\partial y} = \frac{2v-3u}{2(u^2+v^2)}$ and $\frac{\partial v}{\partial y} = \frac{2u+3v}{2(u^2+v^2)}$.

(*b*) Here u and v are to be considered as independent variables. Differentiate the given equations partially with respect to u, obtaining $2u + 2\frac{\partial x}{\partial u} + 3\frac{\partial y}{\partial u} = 0$ and $v + \frac{\partial x}{\partial u} - \frac{\partial y}{\partial u} = 0$. Then $\frac{\partial x}{\partial u} = -\frac{2u+3v}{5}$ and $\frac{\partial y}{\partial u} = \frac{2(v-u)}{5}$.

Differentiate the given equations partially with respect to v, obtaining $-2v + 2\frac{\partial x}{\partial v} + 3\frac{\partial y}{\partial v} = 0$ and $u + \frac{\partial x}{\partial v} - \frac{\partial y}{\partial v} = 0$. Then $\frac{\partial x}{\partial v} = \frac{2v-3u}{5}$ and $\frac{\partial y}{\partial v} = \frac{2u(u+v)}{5}$.

Supplementary Problems

8. Find dy/dx, given

(*a*) $x^3 - x^2y + xy^2 - y^3 = 1$ (*b*) $xy - e^x \sin y = 0$ (*c*) $\ln(x^2+y^2) - \arctan y/x = 0$

Ans. (*a*) $\frac{3x^2-2xy+y^2}{x^2-2xy+3y^2}$; (*b*) $\frac{e^x\sin y - y}{x - e^x\cos y}$; (*c*) $\frac{2x+y}{x-2y}$

9. Find $\partial z/\partial x$ and $\partial z/\partial y$, given

(*a*) $3x^2 + 4y^2 - 5z^2 = 60$ *Ans.* $\partial z/\partial x = 3x/5z$; $\partial z/\partial y = 4y/5z$

(*b*) $x^2 + y^2 + z^2 + 2xy + 4yz + 8zx = 20$ *Ans.* $\frac{\partial z}{\partial x} = -\frac{x+y+4z}{4x+2y+z}$; $\frac{\partial z}{\partial y} = -\frac{x+y+2z}{4x+2y+z}$

(*c*) $x + 3y + 2z = \ln z$ *Ans.* $\frac{\partial z}{\partial x} = \frac{z}{1-2z}$; $\frac{\partial z}{\partial y} = \frac{3z}{1-2z}$

(*d*) $z = e^x\cos(y+z)$ *Ans.* $\frac{\partial z}{\partial x} = \frac{z}{1+e^x\sin(y+z)}$; $\frac{\partial z}{\partial y} = \frac{-e^x\sin(y+z)}{1+e^x\sin(y+z)}$

(*e*) $\sin(x+y) + \sin(y+z) + \sin(z+x) = 1$

Ans. $\frac{\partial z}{\partial x} = -\frac{\cos(x+y)+\cos(z+x)}{\cos(y+z)+\cos(z+x)}$; $\frac{\partial z}{\partial y} = -\frac{\cos(x+y)+\cos(y+z)}{\cos(y+z)+\cos(z+x)}$

10. Find all the first and second partial derivatives of z, given $x^2 + 2yz + 2zx = 1$.

Ans. $\frac{\partial z}{\partial x} = -\frac{x+z}{x+y}$; $\frac{\partial z}{\partial y} = -\frac{z}{x+y}$; $\frac{\partial^2 z}{\partial x^2} = \frac{x-y+2z}{(x+y)^2}$; $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{x+2z}{(x+y)^2}$; $\frac{\partial^2 z}{\partial y^2} = \frac{2z}{(x+y)^2}$

11. If $F(x,y,z) = 0$ show that $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$.

12. If $z = f(x,y)$ and $g(x,y) = 0$, show that $\frac{dz}{dx} = \frac{\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = \frac{1}{\frac{\partial g}{\partial y}} J\left(\frac{f,g}{x,y}\right)$.

13. If $f(x, y) = 0$ and $g(z, x) = 0$, show that $\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}\dfrac{\partial y}{\partial z} = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial z}$.

14. Find the first partial derivatives of u and v with respect to x and y and the first partial derivatives of x and y with respect to u and v, given $2u - v + x^2 + xy = 0$, $u + 2v + xy - y^2 = 0$.

Ans. $\dfrac{\partial u}{\partial x} = -\dfrac{1}{5}(4x + 3y)$; $\dfrac{\partial v}{\partial x} = \dfrac{1}{5}(2x - y)$; $\dfrac{\partial u}{\partial y} = \dfrac{1}{5}(2y - 3x)$; $\dfrac{\partial v}{\partial y} = \dfrac{4y - x}{5}$; $\dfrac{\partial x}{\partial u} = \dfrac{4y - x}{2(x^2 - 2xy - y^2)}$;

$\dfrac{\partial y}{\partial u} = \dfrac{y - 2x}{2(x^2 - 2xy - y^2)}$; $\dfrac{\partial x}{\partial v} = \dfrac{3x - 2y}{2(x^2 - 2xy - y^2)}$; $\dfrac{\partial y}{\partial v} = \dfrac{-4x - 3y}{2(x^2 - 2xy - y^2)}$

15. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, and $w = x^3 + y^3 + z^3$, show that

$$\frac{\partial x}{\partial u} = \frac{yz}{(x - y)(x - z)} \qquad \frac{\partial y}{\partial v} = \frac{x + z}{2(x - y)(y - z)} \qquad \frac{\partial z}{\partial w} = \frac{1}{3(x - z)(y - z)}$$

Chapter 65

Space Vectors

VECTORS IN SPACE. As in the plane (see Chapter 23), a vector in space is a quantity that has both magnitude and direction. Three vectors **a**, **b**, and **c**, not in the same plane and no two parallel, issuing from a common point are said to form a *right-handed system* or *triad* if **c** has the direction in which a right-threaded screw would move when rotated through the smaller angle in the direction from **a** to **b**, as in Fig. 65-1. Note that, as seen from a point on **c**, the rotation through the smaller angle from **a** to **b** is counterclockwise.

We choose a right-handed rectangular coordinate system in space and let **i**, **j**, and **k** be unit vectors along the positive x, y and z axes, respectively, as in Fig. 65-2. The coordinate axes divide space into eight parts, called *octants*. The *first octant*, for example, consists of all points (x, y, z) for which $x > 0$, $y > 0$, $z > 0$.

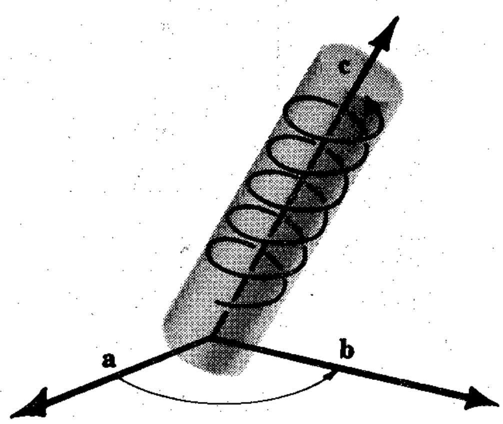

Fig. 65-1

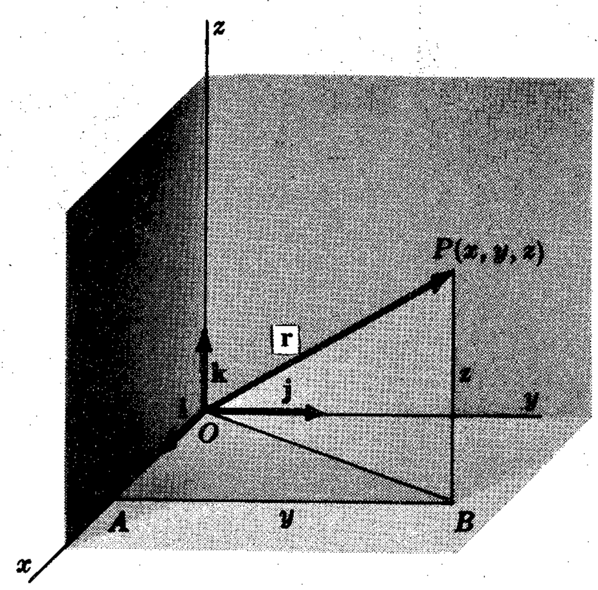

Fig. 65-2

As in Chapter 23, any vector **a** may be written as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

If $P(x, y, z)$ is a point in space (Fig. 65-2), the vector **r** from the origin O to P is called the *position vector* of P and may be written as

$$\mathbf{r} = \mathbf{OP} = \mathbf{OB} + \mathbf{BP} = \mathbf{OA} + \mathbf{AB} + \mathbf{BP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{65.1}$$

The algebra of vectors developed in Chapter 23 holds here with only such changes as the difference in dimensions requires. For example, if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_2\mathbf{k}$, then

$k\mathbf{a} = ka_1\mathbf{i} + ka_2\mathbf{j} + ka_3\mathbf{k}$ for k any scalar

$\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$

$\mathbf{a} \pm \mathbf{b} = (a_1 \pm b_1)\mathbf{i} + (a_2 \pm b_2)\mathbf{j} + (a_3 \pm b_3)\mathbf{k}$

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, where θ is the smaller angle between **a** and **b**

$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$\mathbf{a}\cdot\mathbf{b} = 0$ if $\mathbf{a} = \mathbf{0}$, or $\mathbf{b} = \mathbf{0}$, or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular

From *(65.1)*, we have

$$|\mathbf{r}| = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{x^2 + y^2 + z^2} \tag{65.2}$$

as the distance of the point $P(x, y, z)$ from the origin. Also, if $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points (see Fig. 65-3), then

$$\mathbf{P_1P_2} = \mathbf{P_1B} + \mathbf{BP_2} = \mathbf{P_1A} + \mathbf{AB} + \mathbf{BP_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

and
$$|\mathbf{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{65.3}$$

is the familiar formula for the distance between two points. (See Problems 1 to 3.)

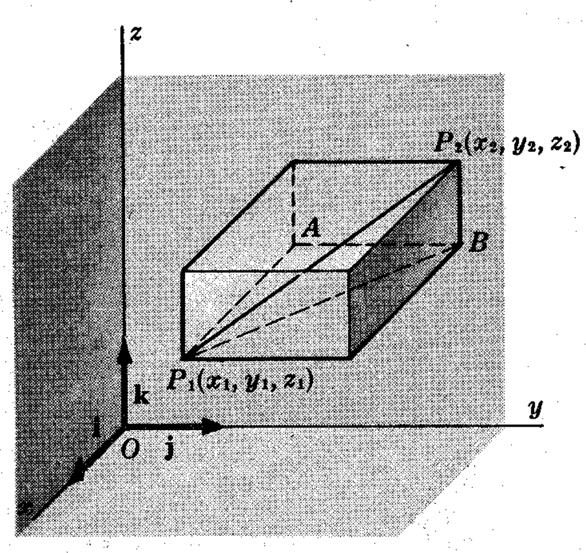

Fig. 65-3

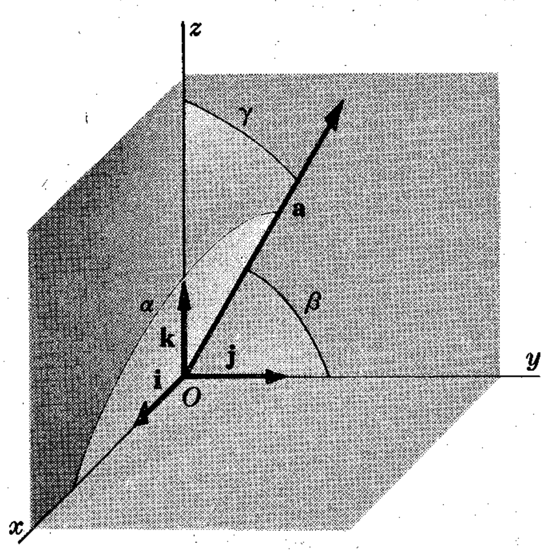

Fig. 65-4

DIRECTION COSINES OF A VECTOR. Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ make angles α, β, and γ, respectively, with the positive x, y, and z axes, as in Fig. 65-4. From

$$\mathbf{i}\cdot\mathbf{a} = |\mathbf{i}||\mathbf{a}|\cos\alpha = |\mathbf{a}|\cos\alpha \qquad \mathbf{j}\cdot\mathbf{a} = |\mathbf{a}|\cos\beta \qquad \mathbf{k}\cdot\mathbf{a} = |\mathbf{a}|\cos\gamma$$

we have

$$\cos\alpha = \frac{\mathbf{i}\cdot\mathbf{a}}{|\mathbf{a}|} = \frac{a_1}{|\mathbf{a}|} \qquad \cos\beta = \frac{\mathbf{j}\cdot\mathbf{a}}{|\mathbf{a}|} = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{\mathbf{k}\cdot\mathbf{a}}{|\mathbf{a}|} = \frac{a_3}{|\mathbf{a}|}$$

These are the *direction cosines* of $\mathbf{a}$. Since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2 + a_2^2 + a_3^2}{|\mathbf{a}|^2} = 1$$

the vector $\mathbf{u} = \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma$ is a unit vector parallel to $\mathbf{a}$.

VECTOR PERPENDICULAR TO TWO VECTORS. Let

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

be two nonparallel vectors with common initial point P. By an easy computation it can be shown that

$$\mathbf{c} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{65.4}$$

is perpendicular to (normal to) both **a** and **b** and, hence, to the plane of these vectors.

In Problems 5 and 6, we show that

$$|\mathbf{c}| = |\mathbf{a}||\mathbf{b}| \sin\theta = \text{area of a parallelogram with nonparallel sides } \mathbf{a} \text{ and } \mathbf{b} \tag{65.5}$$

If **a** and **b** are parallel, then $\mathbf{b} = k\mathbf{a}$, and (*65.4*) shows that $\mathbf{c} = \mathbf{0}$; that is, **c** is the zero vector. The zero vector, by definition, has magnitude 0 but no specified direction.

VECTOR PRODUCT OF TWO VECTORS. Take

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

with initial point P and denote by **n** the unit vector normal to the plane of **a** and **b**, so directed that **a**, **b**, and **n** (in that order) form a right-handed triad at P, as in Fig. 65-5. The *vector product* or *cross product* of **a** and **b** is defined as

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \mathbf{n} \tag{65.6}$$

where θ is again the smaller angle between **a** and **b**. Thus, $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both **a** and **b**.

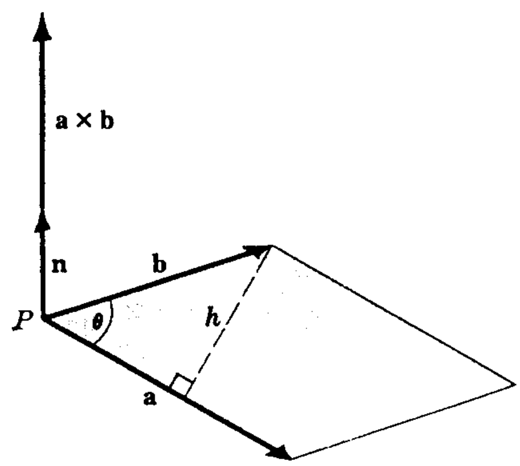

Fig. 65-5

We show in Problem 6 that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ is the area of the parallelogram having **a** and **b** as nonparallel sides.

If **a** and **b** are parallel, then $\theta = 0$ or π and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. Thus,

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0} \tag{65.7}$$

In (*65.6*), if the order of **a** and **b** is reversed, then **n** must be replaced by $-\mathbf{n}$; hence,

$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \tag{65.8}$$

Since the coordinate axes were chosen as a right-handed system, it follows that

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array} \tag{65.9}$$

In Problem 8, we prove for any vectors **a**, **b**, and **c**, the distributive law

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}) \tag{65.10}$$

Multiplying (*65.10*) by -1 and using (*65.8*), we have the companion distributive law

$$\mathbf{c} \times (\mathbf{a} + \mathbf{b}) = (\mathbf{c} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{b}) \tag{65.11}$$

Then, also,

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d} \tag{65.12}$$

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{65.13}$$

(See Problems 9 and 10.)

TRIPLE SCALAR PRODUCT. In Fig. 65-6, let θ be the smaller angle between **b** and **c** and let ϕ be the smaller angle between **a** and $\mathbf{b} \times \mathbf{c}$. Then the triple scalar product is by definition

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{a} \cdot |\mathbf{b}||\mathbf{c}| \sin \theta \, \mathbf{n} = |\mathbf{a}||\mathbf{b}||\mathbf{c}| \sin \theta \cos \phi = (|\mathbf{a}| \cos \phi)(|\mathbf{b}||\mathbf{c}| \sin \theta) = hA \\ &= \text{volume of parallelepiped} \end{aligned}$$

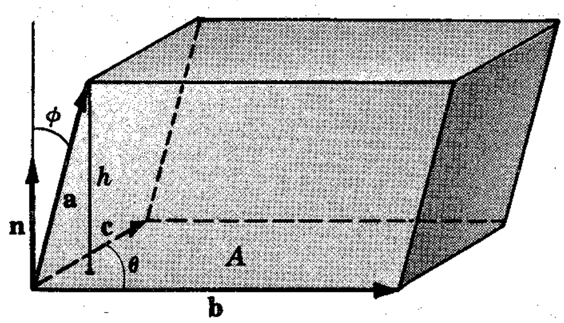

Fig. 65-6

It may be shown (see Problem 11) that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \tag{65.14}$$

Also,

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

while

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Similarly, we have

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \tag{65.15}$$

and

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) \tag{65.16}$$

From the definition of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ as a volume, it follows that if **a**, **b**, and **c** are coplanar, then $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, and conversely.

The parentheses in $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ are not necessary. For example, $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ can be interpreted only as $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ or $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$. But $\mathbf{a} \cdot \mathbf{b}$ is a scalar, so $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ is without meaning. (See Problem 12.)

TRIPLE VECTOR PRODUCT. In Problem 13, we show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \qquad (65.17)$$

Similarly,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \qquad (65.18)$$

Thus, except when $\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{c}$, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and the use of parentheses is necessary.

THE STRAIGHT LINE. A line in space through a given point $P_0(x_0, y_0, z_0)$ may be defined as the locus of all points $P(x, y, z)$ such that P_0P is parallel to a given direction $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P (Fig. 65-7). Then

$$\mathbf{r} - \mathbf{r}_0 = k\mathbf{a} \qquad \text{where } k \text{ is a scalar variable} \qquad (65.19)$$

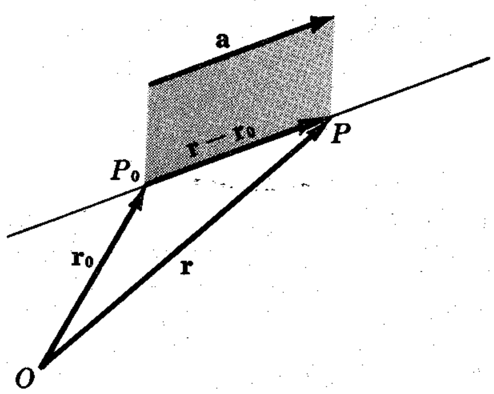

Fig. 65-7

is the vector equation of line PP_0. Writing (65.19) as

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = k(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$$

then separating components to obtain

$$x - x_0 = ka_1 \qquad y - y_0 = ka_2 \qquad z - z_0 = ka_3$$

and eliminating k, we have

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} \qquad (65.20)$$

as the equations of the line in rectangular coordinates. Here, $[a_1, a_2, a_3]$ is a set of *direction numbers* for the line and $\left[\frac{a_1}{|\mathbf{a}|}, \frac{a_2}{|\mathbf{a}|}, \frac{a_3}{|\mathbf{a}|}\right]$ is a set of *direction cosines* of the line.

If any one of the numbers a_1, a_2, a_3 is zero, the corresponding numerator in (65.20) must be zero. For example, if $a_1 = 0$ but $a_2, a_3 \neq 0$, the equations of the line are

$$x - x_0 = 0 \qquad \text{and} \qquad \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}$$

THE PLANE. A plane in space through a given point $P_0(x_0, y_0, z_0)$ can be defined as the locus of all lines through P_0 and a perpendicular (normal) to a given line (direction) $\mathbf{a} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ (Fig. 65-8). Let $P(x, y, z)$ be any other point in the plane. Then $\mathbf{r} - \mathbf{r}_0 = \mathbf{P}_0\mathbf{P}$ is perpendicular to $\mathbf{a}$, and the equation of the plane is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{a} = 0 \qquad (65.21)$$

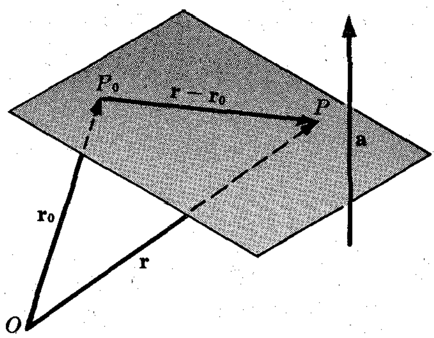

Fig. 65-8

In rectangular coordinates, this becomes

$$[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] \cdot (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$Ax + By + Cz + D = 0 \tag{65.22}$$

where $D = -(Ax_0 + By_0 + Cz_0)$.

Conversely, let $P_0(x_0, y_0, z_0)$ be a point on the surface $Ax + By + Cz + D = 0$. Then also $Ax_0 + By_0 + Cz_0 + D = 0$. Subtracting the second of these equations from the first yields

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

and the constant vector $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the surface at each of its points. Thus, the surface is a plane.

Solved Problems

1. Find the distance of the point $P_1(1, 2, 3)$ from (*a*) the origin, (*b*) the *x* axis, (*c*) the *z* axis, (*d*) the *xy* plane, and (*e*) the point $P_2(3, -1, 5)$.

In Fig. 65-9,

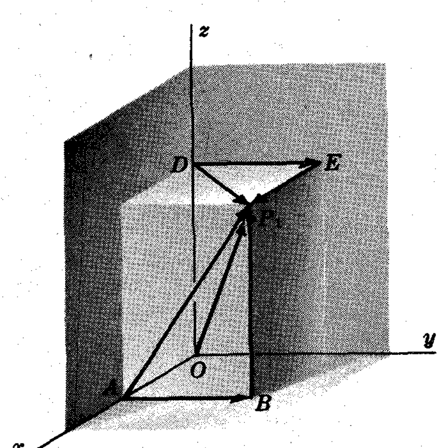

Fig. 65-9

(*a*) $\mathbf{r} = \mathbf{OP}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$; hence, $|\mathbf{r}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.
(*b*) $\mathbf{AP}_1 = \mathbf{AB} + \mathbf{BP}_1 = 2\mathbf{j} + 3\mathbf{k}$; hence, $|\mathbf{AP}_1| = \sqrt{4+9} = \sqrt{13}$.
(*c*) $\mathbf{DP}_1 = \mathbf{DE} + \mathbf{EP}_1 = 2\mathbf{j} + \mathbf{i}$; hence, $|\mathbf{DP}_1| = \sqrt{5}$.
(*d*) $\mathbf{BP}_1 = 3\mathbf{k}$, so $|\mathbf{BP}_1| = 3$.
(*e*) $\mathbf{P}_1\mathbf{P}_2 = (3-1)\mathbf{i} + (-1-2)\mathbf{j} + (5-3)\mathbf{k} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$; hence, $|\mathbf{P}_1\mathbf{P}_2| = \sqrt{4+9+4} = \sqrt{17}$.

2. Find the angle θ between the vectors joining O to $P_1(1,2,3)$ and $P_2(2,-3,-1)$.

Let $\mathbf{r}_1 = \mathbf{OP}_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{r}_2 = \mathbf{OP}_2 = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$. Then

$$\cos\theta = \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{|\mathbf{r}_1||\mathbf{r}_2|} = \frac{1(2) + 2(-3) + 3(-1)}{\sqrt{14}\sqrt{14}} = -\frac{1}{2} \qquad \text{and} \qquad \theta = 120^\circ$$

3. Find the angle $\alpha = \angle BAC$ of the triangle ABC (Fig. 65-10) whose vertices are $A(1,0,1)$, $B(2,-1,1)$, $C(-2,1,0)$.

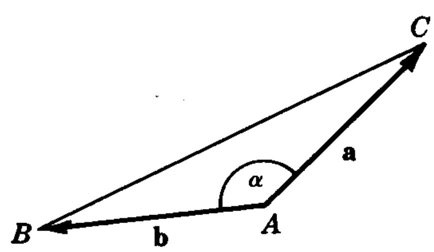

Fig. 65-10

Let $\mathbf{a} = \mathbf{AC} = -3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{AB} = \mathbf{i} - \mathbf{j}$. Then

$$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-3-1}{\sqrt{22}} = -0.85280 \qquad \text{and} \qquad \alpha = 148^\circ 31'$$

4. Find the direction cosines of $\mathbf{a} = 3\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$.

The direction cosines are $\cos\alpha = \dfrac{\mathbf{i} \cdot \mathbf{a}}{|\mathbf{a}|} = \dfrac{3}{13}$, $\cos\beta = \dfrac{\mathbf{j} \cdot \mathbf{a}}{|\mathbf{a}|} = \dfrac{12}{13}$, $\cos\gamma = \dfrac{\mathbf{k} \cdot \mathbf{a}}{|\mathbf{a}|} = \dfrac{4}{13}$.

5. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are two vectors issuing from a point P and if

$$\mathbf{c} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

show that $|\mathbf{c}| = |\mathbf{a}||\mathbf{b}| \sin\theta$, where θ is the smaller angle between $\mathbf{a}$ and $\mathbf{b}$.

We have $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ and

$$\sin\theta = \sqrt{1 - \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)^2} = \frac{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2}}{|\mathbf{a}||\mathbf{b}|} = \frac{|\mathbf{c}|}{|\mathbf{a}||\mathbf{b}|}$$

Hence, $|\mathbf{c}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ as required.

6. Find the area of the parallelogram whose nonparallel sides are $\mathbf{a}$ and $\mathbf{b}$.

From Fig. 65-11, $h = |\mathbf{b}| \sin\theta$ and the area is $h|\mathbf{a}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

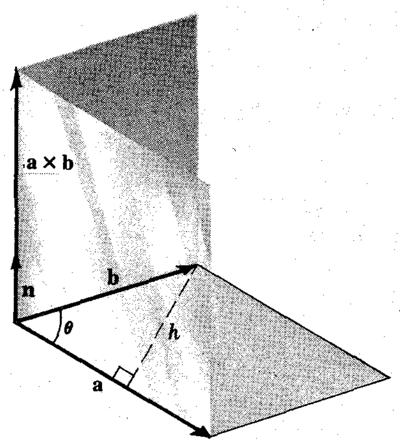

Fig. 65-11

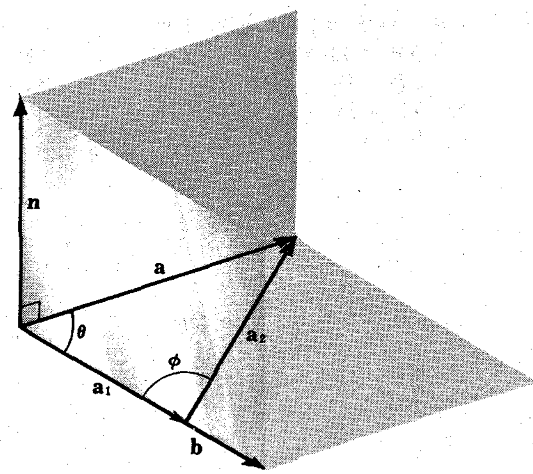

Fig. 65-12

7. Let $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, be the components of $\mathbf{a}$ parallel and perpendicular to $\mathbf{b}$, as in Fig. 65-12. Show that $\mathbf{a}_2 \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{a}_1 \times \mathbf{b} = \mathbf{0}$.

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $|\mathbf{a}_1| = |\mathbf{a}| \cos \theta$ and $|\mathbf{a}_2| = |\mathbf{a}| \sin \theta$. Since $\mathbf{a}$, $\mathbf{a}_2$, and $\mathbf{b}$ are coplanar,

$$\mathbf{a}_2 \times \mathbf{b} = |\mathbf{a}_2||\mathbf{b}| \sin \phi \mathbf{n} = |\mathbf{a}| \sin \theta |\mathbf{b}| \mathbf{n} = |\mathbf{a}||\mathbf{b}| \sin \theta \mathbf{n} = \mathbf{a} \times \mathbf{b}$$

Since $\mathbf{a}_1$ and $\mathbf{b}$ are parallel, $\mathbf{a}_1 \times \mathbf{b} = \mathbf{0}$.

8. Prove: $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$

In Fig. 65-13, the initial point P of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is in the plane of the paper, while their endpoints are above this plane. The vectors $\mathbf{a}_1$ and $\mathbf{b}_1$ are, respectively, the components of $\mathbf{a}$ and $\mathbf{b}$ perpendicular to $\mathbf{c}$. Then $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{a}_1 + \mathbf{b}_1$, $\mathbf{a}_1 \times \mathbf{c}$, $\mathbf{b}_1 \times \mathbf{c}$, and $(\mathbf{a}_1 + \mathbf{b}_1) \times \mathbf{c}$ all lie in the plane of the paper.

In triangles PRS and PMQ,

$$\frac{RS}{PR} = \frac{|\mathbf{b}_1 \times \mathbf{c}|}{|\mathbf{a}_1 \times \mathbf{c}|} = \frac{|\mathbf{b}_1||\mathbf{c}|}{|\mathbf{a}_1||\mathbf{c}|} = \frac{|\mathbf{b}_1|}{|\mathbf{a}_1|} = \frac{MQ}{PM}$$

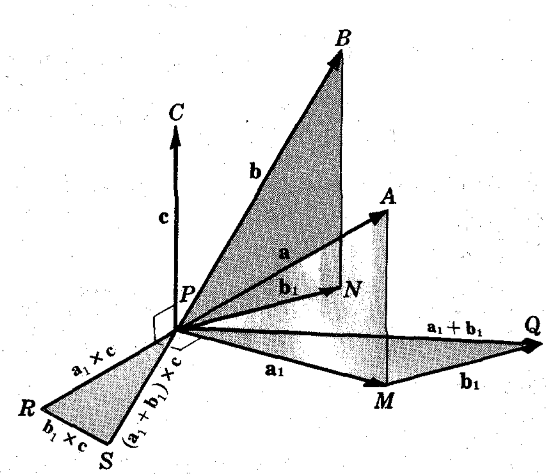

Fig. 65-13

Thus, PRS and PMQ are similar. Now PR is perpendicular to PM, and RS is perpendicular to MQ; hence PS is perpendicular to PQ and $\mathbf{PS} = \mathbf{PQ} \times \mathbf{c}$. Then, since $\mathbf{PS} = \mathbf{PQ} \times \mathbf{c} = \mathbf{PR} + \mathbf{RS}$, we have

$$(\mathbf{a}_1 + \mathbf{b}_1) \times \mathbf{c} = (\mathbf{a}_1 \times \mathbf{c}) + (\mathbf{b}_1 \times \mathbf{c})$$

By Problem 7, $\mathbf{a}_1$ and $\mathbf{b}_1$ may be replaced by $\mathbf{a}$ and $\mathbf{b}$, respectively, to yield the required result.

9. When $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, show that $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

We have, by the distributive law,

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_3\mathbf{k} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= (a_1b_2\mathbf{k} - a_1b_3\mathbf{j}) + (-a_2b_1\mathbf{k} + a_2b_3\mathbf{i}) + (a_3b_1\mathbf{j} - a_3b_2\mathbf{i}) \\
&= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}$$

10. Derive the law of sines of plane trigonometry.

Consider the triangle ABC, whose sides $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are of magnitudes a, b, c, respectively, and whose interior angles are α, β, γ. We have

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$

Then $\quad \mathbf{a} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{0} \quad$ or $\quad \mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$

and $\quad \mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} = \mathbf{0} \quad$ or $\quad \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b}$

Thus, $\quad \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$

so that $\quad |\mathbf{a}||\mathbf{b}| \sin \gamma = |\mathbf{b}||\mathbf{c}| \sin \alpha = |\mathbf{c}||\mathbf{a}| \sin \beta$

or $\quad ab \sin \gamma = bc \sin \alpha = ca \sin \beta$

and $\quad \dfrac{\sin \gamma}{c} = \dfrac{\sin \alpha}{a} = \dfrac{\sin \beta}{b}$

11. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

By (*65.13*),

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot [(b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}] \\
&= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\end{aligned}$$

12. Show that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c}) = 0$.

By (*65.14*), $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{a}) \cdot \mathbf{c} = 0$.

13. For the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ of Problem 11, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Here

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$= (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times[(b_2c_3-b_3c_2)\mathbf{i}+(b_3c_1-b_1c_3)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k}]$$

$$= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1\end{vmatrix}$$

$$= \mathbf{i}(a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3)+\mathbf{j}(a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1)$$
$$+\mathbf{k}(a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2)$$
$$= \mathbf{i}b_1(a_1c_1+a_2c_2+a_3c_3)+\mathbf{j}b_2(a_1c_1+a_2c_2+a_3c_3)+\mathbf{k}b_3(a_1c_1+a_2c_2+a_3c_3)$$
$$-[\mathbf{i}c_1(a_1b_1+a_2b_2+a_3b_3)+\mathbf{j}c_2(a_1b_1+a_2b_2+a_3b_3)+\mathbf{k}c_3(a_1b_1+a_2b_2+a_3b_3)]$$
$$= (b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})(\mathbf{a}\cdot\mathbf{c})-(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})(\mathbf{a}\cdot\mathbf{b})$$
$$= \mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

14. If l_1 and l_2 are two nonintersecting lines in space, show that the shortest distance d between them is the distance from any point on l_1 to the plane through l_2 and parallel to l_1; that is, show that if P_1 is a point on l_1 and P_2 is a point on l_2 then, apart from sign, d is the scalar projection of $\mathbf{P_1P_2}$ on a common perpendicular to l_1 and l_2.

Let l_1 pass through $P_1(x_1, y_1, z_1)$ in the direction $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and let l_2 pass through $P_2(x_2, y_2, z_2)$ in the direction $\mathbf{b} = b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$.

Then $\mathbf{P_1P_2} = (x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$, and the vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to both l_1 and l_2. Thus,

$$d = \left|\frac{\mathbf{P_1P_2}\cdot(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}\right| = \left|\frac{(\mathbf{r}_2-\mathbf{r}_1)\cdot(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}\right|$$

15. Write the equation of the line passing through $P_0(1, 2, 3)$ and parallel to $\mathbf{a} = 2\mathbf{i}-\mathbf{j}-4\mathbf{k}$. Which of the points $A(3, 1, -1)$, $B(1/2, 9/4, 4)$, $C(2, 0, 1)$ are on this line?

From *(65.19)*, the vector equation is

$$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = k(2\mathbf{i}-\mathbf{j}-4\mathbf{k})$$

or

$$(x-1)\mathbf{i}+(y-2)\mathbf{j}+(z-3)\mathbf{k} = k(2\mathbf{i}-\mathbf{j}-4\mathbf{k}) \tag{1}$$

The rectangular equations are

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{-4} \tag{2}$$

Using *(2)*, it is readily found that A and B are on the line while C is not.

In the vector equation *(1)*, a point $P(x, y, z)$ on the line is found by giving k a value and comparing components. The point A is on the line because

$$(3-1)\mathbf{i}+(1-2)\mathbf{j}+(-1-3)\mathbf{k} = k(2\mathbf{i}-\mathbf{j}-4\mathbf{k})$$

when $k = 1$. Similarly B is on the line because

$$-\tfrac{1}{2}\mathbf{i}+\tfrac{1}{4}\mathbf{j}+\mathbf{k} = k(2\mathbf{i}-\mathbf{j}-4\mathbf{k})$$

when $k = -\frac{1}{4}$. The point C is not on the line because

$$\mathbf{i}-2\mathbf{j}-2\mathbf{k} = k(2\mathbf{i}-\mathbf{j}-4\mathbf{k})$$

for no value of k.

16. Write the equation of the plane
(*a*) Passing through $P_0(1,2,3)$ and parallel to $3x-2y+4z-5=0$
(*b*) Passing through $P_0(1,2,3)$ and $P_1(3,-2,1)$, and perpendicular to the plane $3x-2y+4z-5=0$
(*c*) Through $P_0(1,2,3)$, $P_1(3,-2,1)$ and $P_2(5,0,-4)$
Let $P(x, y, z)$ be a general point in the required plane.

(*a*) Here $\mathbf{a}=3\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ is normal to the given plane and to the required plane. The vector equation of the latter is $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{a}=0$ and the rectangular equation is

$$3(x-1)-2(y-2)+4(z-3)=0$$

or

$$3x-2y+4z-11=0$$

(*b*) Here $\mathbf{r}_1-\mathbf{r}_0=2\mathbf{i}-4\mathbf{j}-2\mathbf{k}$ and $\mathbf{a}=3\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ are parallel to the required plane; thus, $(\mathbf{r}_1-\mathbf{r}_0)\times\mathbf{a}$ is normal to this plane. Its vector equation is $(\mathbf{r}-\mathbf{r}_0)\cdot[(\mathbf{r}_1-\mathbf{r}_0)\times\mathbf{a}]=0$. The rectangular equation is

$$(\mathbf{r}-\mathbf{r}_0)\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -4 & -2\\ 3 & -2 & 4\end{vmatrix}=[(x-1)\mathbf{i}+(y-2)\mathbf{j}+(z-3)\mathbf{k}]\cdot[-20\mathbf{i}-14\mathbf{j}+8\mathbf{k}]$$

$$=-20(x-1)-14(y-2)+8(z-3)=0$$

or $20x+14y-8z-24=0$.

(*c*) Here $\mathbf{r}_1-\mathbf{r}_0=2\mathbf{i}-4\mathbf{j}-2\mathbf{k}$ and $\mathbf{r}_2-\mathbf{r}_0=4\mathbf{i}-2\mathbf{j}-7\mathbf{k}$ are parallel to the required plane, so that $(\mathbf{r}_1-\mathbf{r}_0)\times(\mathbf{r}_2-\mathbf{r}_0)$ is normal to it. The vector equation is $(\mathbf{r}-\mathbf{r}_0)\cdot[(\mathbf{r}_1-\mathbf{r}_0)\times(\mathbf{r}_2-\mathbf{r}_0)]=0$ and the rectangular equation is

$$(\mathbf{r}-\mathbf{r}_0)\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -4 & -2\\ 4 & -2 & -7\end{vmatrix}=[(x-1)\mathbf{i}+(y-2)\mathbf{j}+(z-3)\mathbf{k}]\cdot[24\mathbf{i}+6\mathbf{j}+12\mathbf{k}]$$

$$=24(x-1)+6(y-2)+12(z-3)=0$$

or $4x+y+2z-12=0$.

17. Find the shortest distance d between the point $P_0(1,2,3)$ and the plane Π given by the equation $3x-2y+5z-10=0$.

A normal to the plane is $\mathbf{a}=3\mathbf{i}-2\mathbf{j}+5\mathbf{k}$. Take $P_1(2,3,2)$ as a convenient point in Π. Then, apart from sign, d is the scalar projection of $\mathbf{P}_0\mathbf{P}_1$ on $\mathbf{a}$. Hence,

$$d=\left|\frac{(\mathbf{r}_1-\mathbf{r}_0)\cdot\mathbf{a}}{|\mathbf{a}|}\right|=\left|\frac{(\mathbf{i}+\mathbf{j}-\mathbf{k})\cdot(3\mathbf{i}-2\mathbf{j}+5\mathbf{k})}{\sqrt{38}}\right|=\frac{2}{19}\sqrt{38}$$

Supplementary Problems

18. Find the length of (*a*) the vector $\mathbf{a}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}$, (*b*) the vector $\mathbf{b}=3\mathbf{i}-5\mathbf{j}+9\mathbf{k}$, and (*c*) the vector $\mathbf{c}$, joining $P_1(3,4,5)$ to $P_2(1,-2,3)$. *Ans.* (*a*) $\sqrt{14}$, (*b*) $\sqrt{115}$, (*c*) $2\sqrt{11}$

19. For the vectors of Problem 18,
(*a*) Show that **a** and **b** are perpendicular.
(*b*) Find the smaller angle between **a** and **c**, and that between **b** and **c**.
(*c*) Find the angles that **b** makes with the coordinate axes.

Ans. (*b*) 165°14′, 85°10′; (*c*) 73°45′, 117°47′, 32°56′

20. Prove: $\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$ and $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$.

21. Write a unit vector in the direction of **a** and a unit vector in the direction of **b** of Problem 18.

Ans. (*a*) $\frac{\sqrt{14}}{7}\mathbf{i}+\frac{3\sqrt{14}}{14}\mathbf{j}+\frac{\sqrt{14}}{14}\mathbf{k}$; (*b*) $\frac{3}{\sqrt{115}}\mathbf{i}-\frac{5}{\sqrt{115}}\mathbf{j}+\frac{9}{\sqrt{115}}\mathbf{k}$

22. Find the interior angles β and γ of the triangle of Problem 3. *Ans.* $\beta = 22°12'$; $\gamma = 9°16'$

23. For the unit cube in Fig. 65-14, find (*a*) the angle between its diagonal and an edge, and (*b*) the angle between its diagonal and a diagonal of a face.

Ans. (*a*) 54°44′; (*b*) 35°16′

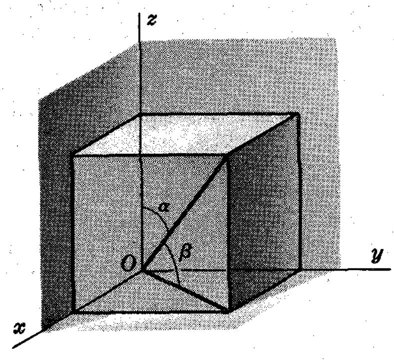

Fig. 65-14

24. Show that the scalar projection of **b** onto **a** is given by $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$.

25. Show that the vector **c** of (*65.4*) is perpendicular to both **a** and **b**.

26. Given $\mathbf{a}=\mathbf{i}+\mathbf{j}$, $\mathbf{b}=\mathbf{i}-2\mathbf{k}$, and $\mathbf{c}=2\mathbf{i}+3\mathbf{j}+4\mathbf{k}$, evaluate the left-hand member:

(*a*) $\mathbf{a}\times\mathbf{b}=-2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ (*b*) $\mathbf{b}\times\mathbf{c}=6\mathbf{i}-8\mathbf{j}+3\mathbf{k}$ (*c*) $\mathbf{c}\times\mathbf{a}=-4\mathbf{i}+4\mathbf{j}-\mathbf{k}$

(*d*) $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=4\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ (*e*) $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=0$ (*f*) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-2$

(*g*) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=3\mathbf{i}-3\mathbf{j}-14\mathbf{k}$ (*h*) $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=-11\mathbf{i}-6\mathbf{j}+10\mathbf{k}$

27. Find the area of the triangle whose vertices are $A(1,2,3)$, $B(2,-1,1)$, and $C(-2,1,-1)$. (*Hint*: $|\mathbf{AB}\times\mathbf{AC}|$ = twice the area.) *Ans.* $5\sqrt{3}$

28. Find the volume of the parallelepiped whose edges are OA, OB, and OC, for $A(1,2,3)$, $B(1,1,2)$, and $C(2,1,1)$. *Ans.* 2

29. If $\mathbf{u}=\mathbf{a}\times\mathbf{b}$, $\mathbf{v}=\mathbf{b}\times\mathbf{c}$, $\mathbf{w}=\mathbf{c}\times\mathbf{a}$, show that

(*a*) $\mathbf{u}\cdot\mathbf{c}=\mathbf{v}\cdot\mathbf{a}=\mathbf{w}\cdot\mathbf{b}$

(*b*) $\mathbf{a}\cdot\mathbf{u}=\mathbf{b}\cdot\mathbf{u}=0$, $\mathbf{b}\cdot\mathbf{v}=\mathbf{c}\cdot\mathbf{v}=0$, $\mathbf{c}\cdot\mathbf{w}=\mathbf{a}\cdot\mathbf{w}=0$

(*c*) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2$

30. Show that $(\mathbf{a}+\mathbf{b})\cdot[(\mathbf{b}+\mathbf{c})\times(\mathbf{c}+\mathbf{a})]=2\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

31. Find the smaller angle of intersection of the planes $5x-14y+2z-8=0$ and $10x-11y+2z+15=0$. (*Hint*: Find the angle between their normals.) *Ans.* 22°25′

32. Write the vector equation of the line of intersection of the planes $x + y - z - 5 = 0$ and $4x - y - z + 2 = 0$. *Ans.* $(x-1)\mathbf{i} + (y-5)\mathbf{j} + (z-1)\mathbf{k} = k(-2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$, where $P_0(1, 5, 1)$ is a point on the line

33. Find the shortest distance between the line through $A(2, -1, -1)$ and $B(6, -8, 0)$ and the line through $C(2, 1, 2)$ and $D(0, 2, -1)$. *Ans.* $\sqrt{6}/6$

34. Define a line through $P_0(x_0, y_0, z_0)$ as the locus of all points $P(x, y, z)$ such that $\mathbf{P_0P}$ and $\mathbf{OP_0}$ are perpendicular. Show that its vector equation is $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{r}_0 = 0$.

35. Find the rectangular equations of the line through $P_0(2, -3, 5)$ and
(*a*) Perpendicular to $7x - 4y + 2z - 8 = 0$
(*b*) Parallel to the line $x - y + 2z + 4 = 0$, $2x + 3y + 6z - 12 = 0$
(*c*) Through $P_1(3, 6, -2)$

Ans. (*a*) $\dfrac{x-2}{7} = \dfrac{y+3}{-4} = \dfrac{z-5}{2}$; (*b*) $\dfrac{x-2}{12} = \dfrac{y+3}{2} = \dfrac{z-5}{-5}$; (*c*) $\dfrac{x-2}{1} = \dfrac{y+3}{9} = \dfrac{z-5}{-7}$

36. Find the equation of the plane
(*a*) Through $P_0(1, 2, 3)$ and parallel to $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$
(*b*) Through $P_0(2, -3, 2)$ and the line $6x + 4y + 3z + 5 = 0$, $2x + y + z - 2 = 0$
(*c*) Through $P_0(2, -1, -1)$ and $P_1(1, 2, 3)$ and perpendicular to $2x + 3y - 5z - 6 = 0$

Ans. (*a*) $4x + y + 9z - 33 = 0$; (*b*) $16x + 7y + 8z - 27 = 0$; (*c*) $9x - y + 3z - 16 = 0$

37. If $r_0 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $r_1 = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and $r_2 = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$ are three position vectors, show that $\mathbf{r}_0 \times \mathbf{r}_1 + \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_0 = \mathbf{0}$. What can be said of the terminal points of these vectors? *Ans.* collinear

38. If P_0, P_1, and P_2 are three noncollinear points and $\mathbf{r}_0$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are their position vectors, what is the position of $\mathbf{r}_0 \times \mathbf{r}_1 + \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_0$ with respect to the plane $P_0P_1P_2$? *Ans.* normal

39. Prove: (*a*) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$
(*b*) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$

40. Prove: (*a*) The perpendiculars erected at the midpoints of the sides of a triangle meet in a point.
(*b*) The perpendiculars dropped from the vertices to the opposite sides (produced if necessary) of a triangle meet in a point.

41. Let $A(1, 2, 3)$, $B(2, -1, 5)$, and $C(4, 1, 3)$ be three vertices of the parallelogram $ABCD$. Find (*a*) the coordinates of D, (*b*) the area of $ABCD$, and (*c*) the area of the orthogonal projection of $ABCD$ on each of the coordinate planes. *Ans.* (*a*) $D(3, 4, 1)$; (*b*) $2\sqrt{26}$; (*c*) 8, 6, 2

42. Prove that the area of a parallelogram in space is the square root of the sum of the squares of the areas of projections of the parallelogram on the coordinate planes.

Chapter 66

Space Curves and Surfaces

TANGENT LINE AND NORMAL PLANE TO A SPACE CURVE. A space curve may be defined parametrically by the equations

$$x = f(t) \qquad y = g(t) \qquad z = h(t) \tag{66.1}$$

At the point $P_0(x_0, y_0, z_0)$ of the curve (determined by $t = t_0$), the equations of the *tangent line* are

$$\frac{x - x_0}{dx/dt} = \frac{y - y_0}{dy/dt} = \frac{z - z_0}{dz/dt} \tag{66.2}$$

and the equation of the *normal plane* (the plane through P_0 perpendicular to the tangent line there) is

$$\frac{dx}{dt}(x - x_0) + \frac{dy}{dt}(y - y_0) + \frac{dz}{dt}(z - z_0) = 0 \tag{66.3}$$

(See Fig. 66-1.) In both (*66.2*) and (*66.3*) it is understood that the derivatives have been evaluated at the point P_0. (See Problems 1 and 2.)

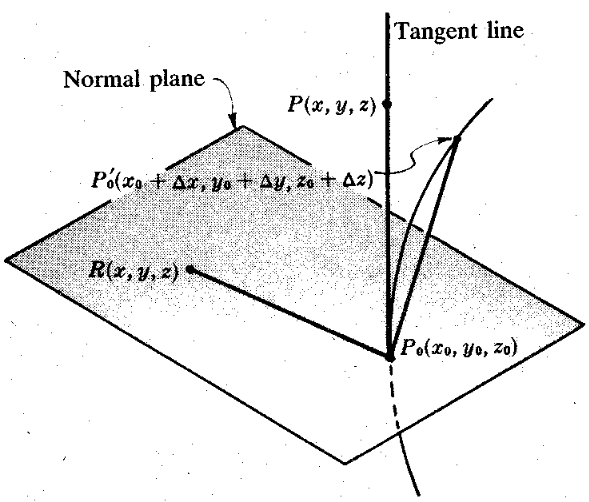

Fig. 66-1

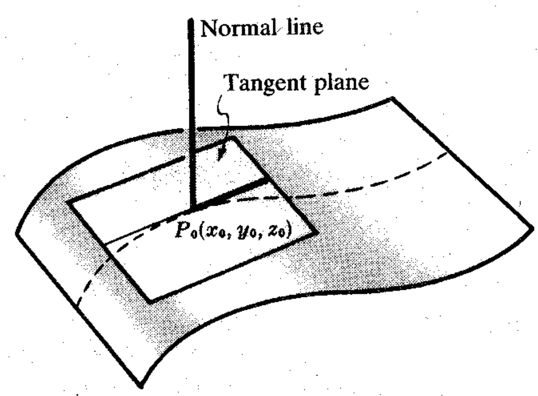

Fig. 66-2

TANGENT PLANE AND NORMAL LINE TO A SURFACE. The equation of the *tangent plane* to the surface $F(x, y, z) = 0$ at one of its points $P_0(x_0, y_0, z_0)$ is

$$\frac{\partial F}{\partial x}(x - x_0) + \frac{\partial F}{\partial y}(y - y_0) + \frac{\partial F}{\partial z}(z - z_0) = 0 \tag{66.4}$$

and the equations of the *normal line* at P_0 are

$$\frac{x - x_0}{\partial F/\partial x} = \frac{y - y_0}{\partial F/\partial y} = \frac{z - z_0}{\partial F/\partial z} \tag{66.5}$$

with the understanding that the partial derivatives have been evaluated at the point P_0. (Refer to Fig. 66-2.) (See Problems 3 to 9.)

A SPACE CURVE may also be defined by a pair of equations

$$F(x, y, z) = 0 \qquad G(x, y, z) = 0 \tag{66.6}$$

At the point $P_0(x_0, y_0, z_0)$ of the curve, the equations of the tangent line are

$$\frac{x - x_0}{\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}} = \frac{y - y_0}{\begin{vmatrix} \frac{\partial F}{\partial z} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial x} \end{vmatrix}} = \frac{z - z_0}{\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix}} \tag{66.7}$$

and the equation of the normal plane is

$$\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}(x - x_0) + \begin{vmatrix} \frac{\partial F}{\partial z} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial z} & \frac{\partial G}{\partial x} \end{vmatrix}(y - y_0) + \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{vmatrix}(z - z_0) = 0 \tag{66.8}$$

In (*66.7*) and (*66.8*) it is to be understood that all partial derivatives have been evaluated at the point P_0. (See Problems 10 and 11.)

Solved Problems

1. Derive (*66.2*) and (*66.3*) for the tangent line and normal plane to the space curve $x = f(t)$, $y = g(t)$, $z = h(t)$ at the point $P_0(x_0, y_0, z_0)$ determined by the value $t = t_0$. Refer to Fig. 66-1.

Let $P_0'(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$, determined by $t = t_0 + \Delta t$, be another point on the curve. As $P_0 \to P_0$ along the curve, the chord P_0P_0 approaches the tangent line to the curve at P_0 as limiting position.

A simple set of direction numbers for the chord P_0P_0 is $[\Delta x, \Delta y, \Delta z]$, but we shall use $\left[\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right]$. Then as $P_0 \to P_0$, $\Delta t \to 0$ and $\left[\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right] \to \left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right]$, a set of direction numbers of the tangent line at P_0. Now if $P(x, y, z)$ is an arbitrary point on this tangent line, then $[x - x_0, y - y_0, z - z_0]$ is a set of direction numbers of P_0P. Thus, since the sets of direction numbers are proportional, the equations of the tangent line at P_0 are

$$\frac{x - x_0}{dx/dt} = \frac{y - y_0}{dy/dt} = \frac{z - z_0}{dz/dt}$$

If $R(x, y, z)$ is an arbitrary point in the normal plane at P_0 then, since P_0R and P_0P are perpendicular, the equation of the normal plane at P_0 is

$$(x - x_0)\frac{dx}{dt} + (y - y_0)\frac{dy}{dt} + (z - z_0)\frac{dz}{dt} = 0$$

2. Find the equations of the tangent line and normal plane to
(*a*) The curve $x = t$, $y = t^2$, $z = t^3$ at the point $t = 1$
(*b*) The curve $x = t - 2$, $y = 3t^2 + 1$, $z = 2t^3$ at the point where it pierces the yz plane.

(*a*) At the point $t = 1$ or $(1, 1, 1)$, $dx/dt = 1$, $dy/dt = 2t = 2$, and $dz/dt = 3t^2 = 3$. Using (*66.2*) yields, for the equations of the tangent line, $\frac{x - 1}{1} = \frac{y - 1}{2} = \frac{z - 1}{3}$; using (*66.3*) gives the equation of the normal plane as $(x - 1) + 2(y - 1) + 3(z - 1) = x + 2y + 3z - 6 = 0$.

(*b*) The given curve pierces the yz plane at the point where $x = t - 2 = 0$, that is, at the point $t = 2$ or $(0, 13, 16)$. At this point, $dx/dt = 1$, $dy/dt = 6t = 12$, and $dz/dt = 6t^2 = 24$. The equations of the tangent line are $\frac{x}{1} = \frac{y - 13}{12} = \frac{z - 16}{24}$, and the equation of the normal plane is $x + 12(y - 13) + 24(z - 16) = x + 12y + 24z - 540 = 0$.

3. Derive (*66.4*) and (*66.5*) for the tangent plane and normal line to the surface $F(x, y, z) = 0$ at the point $P_0(x_0, y_0, z_1)$. Refer to Fig. 66-2.

Let $x = f(t)$, $y = g(t)$, $z = h(t)$ be the parametric equations of any curve on the surface $F(x, y, z) = 0$ and passing through the point P_0. Then, at P_0,

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

with the understanding that all derivatives have been evaluated at P_0.

This relation expresses the fact that the line through P_0 with direction numbers $\left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right]$ is perpendicular to the line through P_0 having direction numbers $\left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right]$. The first set of direction numbers belongs to the tangent to the curve which lies in the tangent plane of the surface. The second set defines the normal line to the surface at P_0. The equations of this normal are

$$\frac{x - x_0}{\partial F/\partial x} = \frac{y - y_0}{\partial F/\partial y} = \frac{z - z_0}{\partial F/\partial z}$$

and the equation of the tangent plane at P_0 is

$$\frac{\partial F}{\partial x}(x - x_0) + \frac{\partial F}{\partial y}(y - y_0) + \frac{\partial F}{\partial z}(z - z_0) = 0$$

In Problems 4 and 5, find the equations of the tangent plane and normal line to the given surface at the given point.

4. $z = 3x^2 + 2y^2 - 11$; $(2, 1, 3)$

Put $F(x, y, z) = 3x^2 + 2y^2 - z - 11 = 0$. At $(2, 1, 3)$, $\frac{\partial F}{\partial x} = 6x = 12$, $\frac{\partial F}{\partial y} = 4y = 4$, and $\frac{\partial F}{\partial z} = -1$. The equation of the tangent plane is $12(x - 2) + 4(y - 1) - (z - 3) = 0$ or $12x + 4y - z = 25$.

The equations of the normal line are $\frac{x-2}{12} = \frac{y-1}{4} = \frac{z-3}{-1}$.

5. $F(x, y, z) = x^2 + 3y^2 - 4z^2 + 3xy - 10yz + 4x - 5z - 22 = 0$; $(1, -2, 1)$

At $(1, -2, 1)$, $\frac{\partial F}{\partial x} = 2x + 3y + 4 = 0$, $\frac{\partial F}{\partial y} = 6y + 3x - 10z = -19$, and $\frac{\partial F}{\partial z} = -8z - 10y - 5 = 7$. The equation of the tangent plane is $0(x - 1) - 19(y + 2) + 7(z - 1) = 0$ or $19y - 7z + 45 = 0$.

The equations of the normal line are $x - 1 = 0$ and $\frac{y+2}{-19} = \frac{z-1}{7}$ or $x = 1$, $7y + 19z - 5 = 0$.

6. Show that the equation of the tangent plane to the surface $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ at the point $P_0(x_0, y_0, z_0)$ is $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = 1$.

At P_0, $\frac{\partial F}{\partial x} = \frac{2x_0}{a^2}$, $\frac{\partial F}{\partial y} = -\frac{2y_0}{b^2}$, and $\frac{\partial F}{\partial z} = -\frac{2z_0}{c^2}$. The equation of the tangent plane is $\frac{2x_0}{a^2}(x - x_0) - \frac{2y_0}{b^2}(y - y_0) - \frac{2z_0}{c^2}(z - z_0) = 0$.

This becomes $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} = 1$, since P_0 is on the surface.

7. Show that the surfaces

$$F(x, y, z) = x^2 + 4y^2 - 4z^2 - 4 = 0 \quad \text{and} \quad G(x, y, z) = x^2 + y^2 + z^2 - 6x - 6y + 2z + 10 = 0$$

are tangent at the point $(2, 1, 1)$.

It is to be shown that the two surfaces have the same tangent plane at the given point. At $(2, 1, 1)$,

$$\frac{\partial F}{\partial x} = 2x = 4 \qquad \frac{\partial F}{\partial y} = 8y = 8 \qquad \frac{\partial F}{\partial z} = -8z = -8$$

and
$$\frac{\partial G}{\partial x} = 2x - 6 = -2 \qquad \frac{\partial G}{\partial y} = 2y - 6 = -4 \qquad \frac{\partial G}{\partial z} = 2z + 2 = 4$$

Since the sets of direction numbers $[4, 8, -8]$ and $[-2, -4, 4]$ of the normal lines of the two surfaces are proportional, the surfaces have the common tangent plane

$$1(x-2) + 2(y-1) - 2(z-1) = 0 \qquad \text{or} \qquad x + 2y - 2z = 2$$

8. Show that the surfaces $F(x, y, z) = xy + yz - 4zx = 0$ and $G(x, y, z) = 3z^2 - 5x + y = 0$ intersect at right angles at the point $(1, 2, 1)$.

It is to be shown that the tangent planes to the surfaces at the point are perpendicular or, what is the same, that the normal lines at the point are perpendicular. At $(1, 2, 1)$,

$$\frac{\partial F}{\partial x} = y - 4z = -2 \qquad \frac{\partial F}{\partial y} = x + z = 2 \qquad \frac{\partial F}{\partial z} = y - 4x = -2$$

A set of direction numbers for the normal line to $F(x, y, z) = 0$ is $[l_1, m_1, n_1] = [1, -1, 1]$. At the same point,

$$\frac{\partial G}{\partial x} = -5 \qquad \frac{\partial G}{\partial y} = 1 \qquad \frac{\partial G}{\partial z} = 6z = 6$$

A set of direction numbers for the normal line to $G(x, y, z) = 0$ is $[l_2, m_2, n_2] = [-5, 1, 6]$.
Since $l_1 l_2 + m_1 m_2 + n_1 n_2 = 1(-5) + (-1)1 + 1(6) = 0$, these directions are perpendicular.

9. Show that the surfaces $F(x, y, z) = 3x^2 + 4y^2 + 8z^2 - 36 = 0$ and $G(x, y, z) = x^2 + 2y^2 - 4z^2 - 6 = 0$ intersect at right angles.

At any point $P_0(x_0, y_0, z_0)$ on the two surfaces, $\frac{\partial F}{\partial x} = 6x_0$, $\frac{\partial F}{\partial y} = 8y_0$, and $\frac{\partial F}{\partial z} = 16z_0$; hence $[3x_0, 4y_0, 8z_0]$ is a set of direction numbers for the normal to the surface $F(x, y, z) = 0$ at P_0. Similarly, $[x_0, 2y_0, -4z_0]$ is a set of direction numbers for the normal line to $G(x, y, z) = 0$ at P_0. Now, since

$$\begin{aligned} 3x_0(x_0) + 4y_0(2y_0) + 8z_0(-4z_0) &= 3x_0^2 + 8y_0^2 - 32z_0^2 \\ &= 6(x_0^2 + 2y_0^2 - 4z_0^2) - (3x_0^2 + 4y_0^2 + 8z_0^2) = 6(6) - 36 = 0 \end{aligned}$$

these directions are perpendicular.

10. Derive *(66.7)* and *(66.8)* for the tangent line and normal plane to the space curve C: $F(x, y, z) = 0$, $G(x, y, z) = 0$ at one of its points $P_0(x_0, y_0, z_0)$.

At P_0 the directions $\left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right]$ and $\left[\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\right]$ are normal, respectively, to the tangent planes of the surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$. Now the direction

$$\left[\begin{vmatrix} \partial F/\partial y & \partial F/\partial z \\ \partial G/\partial y & \partial G/\partial z \end{vmatrix}, \begin{vmatrix} \partial F/\partial z & \partial F/\partial x \\ \partial G/\partial z & \partial G/\partial x \end{vmatrix}, \begin{vmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{vmatrix}\right]$$

being perpendicular to each of these directions, is that of the tangent line to C at P_0. Hence, the equations of the tangent line are

$$\frac{x - x_0}{\begin{vmatrix} \partial F/\partial y & \partial F/\partial z \\ \partial G/\partial y & \partial G/\partial z \end{vmatrix}} = \frac{y - y_0}{\begin{vmatrix} \partial F/\partial z & \partial F/\partial x \\ \partial G/\partial z & \partial G/\partial x \end{vmatrix}} = \frac{z - z_0}{\begin{vmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{vmatrix}}$$

and the equation of the normal plane is

$$\begin{vmatrix} \partial F/\partial y & \partial F/\partial z \\ \partial G/\partial y & \partial G/\partial z \end{vmatrix}(x - x_0) + \begin{vmatrix} \partial F/\partial z & \partial F/\partial x \\ \partial G/\partial z & \partial G/\partial x \end{vmatrix}(y - y_0) + \begin{vmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{vmatrix}(z - z_0) = 0$$

11. Find the equations of the tangent line and the normal plane to the curve $x^2 + y^2 + z^2 = 14$, $x + y + z = 6$ at the point $(1, 2, 3)$.

Set $F(x, y, z) = x^2 + y^2 + z^2 - 14 = 0$ and $G(x, y, z) = x + y + z - 6 = 0$. At $(1, 2, 3)$,

$$\begin{vmatrix} \partial F/\partial y & \partial F/\partial z \\ \partial G/\partial y & \partial G/\partial z \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 1 & 1 \end{vmatrix} = -2$$

$$\begin{vmatrix} \partial F/\partial z & \partial F/\partial x \\ \partial G/\partial z & \partial G/\partial x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 1 & 1 \end{vmatrix} = 4 \qquad \begin{vmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = -2$$

With $[1, -2, 1]$ as a set of direction numbers of the tangent, its equations are $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{1}$. The equation of the normal plane is $(x - 1) - 2(y - 2) + (z - 3) = x - 2y + z = 0$.

Supplementary Problems

12. Find the equations of the tangent line and the normal plane to the given curve at the given point:

(*a*) $x = 2t,\ y = t^2,\ z = t^3;\ t = 1$ *Ans.* $\frac{x-2}{2} = \frac{y-1}{2} = \frac{z-1}{3}$; $2x + 2y + 3z - 9 = 0$

(*b*) $x = te^t,\ y = e^t,\ z = t;\ t = 0$ *Ans.* $\frac{x}{1} = \frac{y-1}{1} = \frac{z}{1}$; $x + y + z - 1 = 0$

(*c*) $x = t\cos t,\ y = t\sin t,\ z = t;\ t = 0$ *Ans.* $x = z,\ y = 0$; $x + z = 0$

13. Show that the curves (*a*) $x = 2 - t,\ y = -1/t,\ z = 2t^2$ and (*b*) $x = 1 + \theta,\ y = \sin\theta - 1,\ z = 2\cos\theta$ intersect at right angles at $P(1, -1, 2)$. Obtain the equations of the tangent line and normal plane of each curve at P.

Ans. (*a*) $\frac{x-1}{-1} = \frac{y+1}{1} = \frac{z-2}{4}$; $x - y - 4z + 6 = 0$; (*b*) $x - y = 2,\ z = 2$; $x + y = 0$

14. Show that the tangents to the helix $x = a\cos t,\ y = a\sin t,\ z = bt$ meet the xy plane at the same angle.

15. Show that the length of the curve (*66.1*) from the point $t = t_0$ to the point $t = t_1$ is given by

$$\int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Find the length of the helix of Problem 14 from $t = 0$ to $t = t_1$. *Ans.* $\sqrt{a^2 + b^2}\, t_1$

16. Find the equations of the tangent line and the normal plane to the given curve at the given point:
(*a*) $x^2 + 2y^2 + 2z^2 = 5,\ 3x - 2y - z = 0;\ (1, 1, 1)$
(*b*) $9x^2 + 4y^2 - 36z = 0,\ 3x + y + z - z^2 - 1 = 0;\ (2, -3, 2)$
(*c*) $4z^2 = xy,\ x^2 + y^2 = 8z;\ (2, 2, 1)$

Ans. (*a*) $\frac{x-1}{2} = \frac{y-1}{7} = \frac{z-1}{-8}$; $2x + 7y - 8z - 1 = 0$; (*b*) $\frac{x-2}{1} = \frac{z-2}{1}$, $y + 3 = 0$; $x + z - 4 = 0$;

(*c*) $\frac{x-2}{1} = \frac{y-2}{-1}$, $z - 1 = 0$; $x - y = 0$

17. Find the equations of the tangent plane and normal line to the given surface at the given point:

(*a*) $x^2 + y^2 + z^2 = 14;\ (1, -2, 3)$ *Ans.* $x - 2y + 3z = 14$; $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3}$

(*b*) $x^2 + y^2 + z^2 = r^2;\ (x_1, y_1, z_1)$ *Ans.* $x_1x + y_1y + z_1z = r^2$; $\frac{x-x_1}{x_1} = \frac{y-y_1}{y_1} = \frac{z-z_1}{z_1}$

(*c*) $x^2 + 2z^2 = 3y^2$; $(2, -2, -2)$ *Ans.* $x + 3y - 2z = 0; \dfrac{x-2}{1} = \dfrac{y+2}{3} = \dfrac{z+2}{-2}$

(*d*) $2x^2 + 2xy + y^2 + z + 1 = 0$; $(1, -2, -3)$ *Ans.* $z - 2y = 1; x - 1 = 0, \dfrac{y+2}{2} = \dfrac{z+3}{-1}$

(*e*) $z = xy$; $(3, -4, -12)$ *Ans.* $4x - 3y + z = 12; \dfrac{x-3}{4} = \dfrac{y+4}{-3} = \dfrac{z+12}{1}$

18. (*a*) Show that the sum of the intercepts of the plane tangent to the surface $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$ at any of its points is a.
(*b*) Show that the square root of the sum of the squares of the intercepts of the plane tangent to the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ at any of its points is a.

19. Show that each pair of surfaces is tangent at the given point:
(*a*) $x^2 + y^2 + z^2 = 18$, $xy = 9$; $(3, 3, 0)$
(*b*) $x^2 + y^2 + z^2 - 8x - 8y - 6z + 24 = 0$, $x^2 + 3y^2 + 2z^2 = 9$; $(2, 1, 1)$

20. Show that each pair of surfaces is mutually perpendicular at the given point:
(*a*) $x^2 + 2y^2 - 4z^2 = 8$, $4x^2 - y^2 + 2z^2 = 14$; $(2, 2, 1)$
(*b*) $x^2 + y^2 + z^2 = 50$, $x^2 + y^2 - 10z + 25 = 0$; $(3, 4, 5)$

21. Show that each of the surfaces (*a*) $14x^2 + 11y^2 + 8z^2 = 66$, (*b*) $3z^2 - 5x + y = 0$, and (*c*) $xy + yz - 4zx = 0$ is perpendicular to the other two at the point $(1, 2, 1)$.

Chapter 67

Directional Derivatives; Maximum and Minimum Values

DIRECTIONAL DERIVATIVES. Through $P(x, y, z)$, any point on the surface $z = f(x, y)$, pass planes parallel to the coordinate planes xOz and yOz cutting the surface in the arcs PR and PS and the plane xOy in the lines P^*M and P^*N, as shown in Fig. 67-1. The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ evaluated at $P^*(x, y)$ give, respectively the rates of change of $z = P^*P$ when y is held fixed and when x is held fixed, that is, the rates of change of z in directions parallel to the x and y axes or the slopes of the curves PR and PS at P.

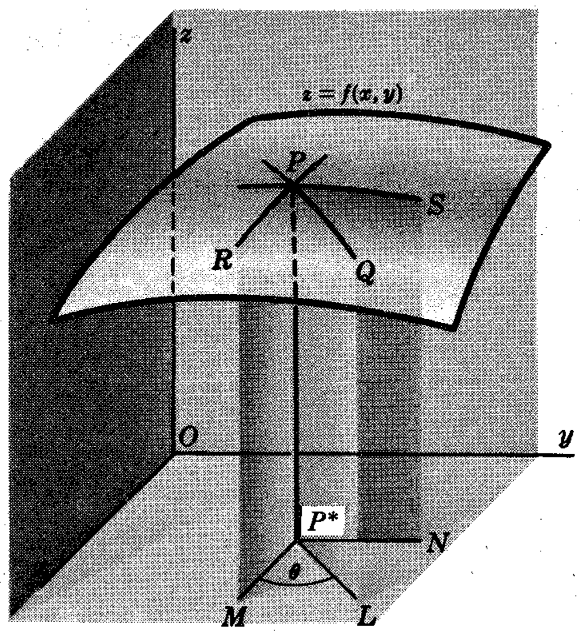

Fig. 67-1

Consider next a plane through P perpendicular to the plane xOy and making an angle θ with the x axis. Let it cut the surface in the curve PQ and the xOy plane in the line P^*L. The *directional derivative* of $f(x, y)$ at P^* in the direction θ is given by

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \qquad (67.1)$$

The direction θ is the direction of the vector $(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. The directional derivative gives the rate of change of $z = P^*P$ in the direction of P^*L or the slope of the curve PQ at P.

The directional derivative at a point P^* is a function of θ. There is a direction, determined by a vector called the *gradient* of f at P^* (Chapter 68), for which the directional derivative at P^* has a maximum value. That maximum value is the slope of the steepest tangent line that can be drawn to the surface at P. (See Problems 1 to 8.)

For a function $w = F(x, y, z)$, the directional derivative at $P(x, y, z)$ in the direction determined by the angles α, β, γ is given by

$$\frac{dF}{ds} = \frac{\partial F}{\partial x}\cos\alpha + \frac{\partial F}{\partial y}\cos\beta + \frac{\partial F}{\partial z}\cos\gamma$$

By the direction determined by α, β, and γ, we mean the direction of the vector $(\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}$. (See Problem 9.)

RELATIVE MAXIMUM AND MINIMUM VALUES. Suppose that $z = f(x, y)$ has a relative maximum (or minimum) value at $P_0(x_0, y_0, z_0)$. Any plane through P_0 perpendicular to the plane xOy will cut the surface in a curve having a relative maximum (or minimum) point at P_0; that is, the directional derivative $\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$ of $z = f(x, y)$ must equal zero at P_0, for any value of θ. Thus, at P_0, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

The points, if any, at which $z = f(x, y)$ has a relative maximum (or minimum) value are among the points (x_0, y_0) for which $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ simultaneously. To separate the cases, we quote without proof:

Let $z = f(x, y)$ have first and second partial derivatives in a certain region including the point (x_0, y_0, z_0) at which $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. If $\Delta = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) < 0$ at P_0, then $z = f(x, y)$ has

$$\text{A relative minimum at } P_0 \text{ if } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} > 0$$

or

$$\text{A relative maximum at } P_0 \text{ if } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} < 0$$

If $\Delta > 0$, P_0 yields neither a maximum nor a minimum value; if $\Delta = 0$, the nature of the critical point P_0 is undetermined. (See Problems 10 to 15.)

Solved Problems

1. Derive (*67.1*).

In Fig. 67-1, let $P_1^*(x + \Delta x, y + \Delta y)$ be a second point on P^*L and denote by Δs the distance $P^*P_1^*$. Assuming that $z = f(x, y)$ possesses continuous first partial derivatives, we have, by Problem 20 of Chapter 63,

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as Δx and $\Delta y \to 0$. The average rate of change of z between the points P^* and P_1^* is

$$\frac{\Delta z}{\Delta s} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta s} + \epsilon_1\frac{\Delta x}{\Delta s} + \epsilon_2\frac{\Delta y}{\Delta s}$$

$$= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta + \epsilon_1\cos\theta + \epsilon_2\sin\theta$$

where θ is the angle that the line $P^*P_1^*$ makes with the x axis. Now let $P_1^* \to P^*$ along P^*L; the instantaneous rate of change of z, or the directional derivative at P^*, is

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

2. Find the directional derivative of $z = x^2 - 6y^2$ at $P^*(7, 2)$ in the direction (*a*) $\theta = 45°$, (*b*) $\theta = 135°$.

The directional derivative at any point $P^*(x, y)$ in the direction θ is

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta = 2x\cos\theta - 12y\sin\theta$$

(*a*) At $P^*(7, 2)$ in the direction $\theta = 45°$, $dz/ds = 2(7)(\frac{1}{2}\sqrt{2}) - 12(2)(\frac{1}{2}\sqrt{2}) = -5\sqrt{2}$.
(*b*) At $P^*(7, 2)$ in the direction $\theta = 135°$, $dz/ds = 2(7)(-\frac{1}{2}\sqrt{2}) - 12(2)(\frac{1}{2}\sqrt{2}) = -19\sqrt{2}$.

3. Find the directional derivative of $z = ye^x$ at $P^*(0, 3)$ in the direction (*a*) $\theta = 30°$, (*b*) $\theta = 120°$.

Here, $dz/ds = ye^x \cos\theta + e^x \sin\theta$.
(*a*) At $(0, 3)$ in the direction $\theta = 30°$, $dz/ds = 3(1)(\frac{1}{2}\sqrt{3}) + \frac{1}{2} = \frac{1}{2}(3\sqrt{3} + 1)$.
(*b*) At $(0, 3)$ in the direction $\theta = 120°$, $dz/ds = 3(1)(-\frac{1}{2}) + \frac{1}{2}\sqrt{3} = \frac{1}{2}(-3 + \sqrt{3})$.

4. The temperature T of a heated circular plate at any of its points (x, y) is given by $T = \dfrac{64}{x^2 + y^2 + 2}$, the origin being at the center of the plate. At the point $(1, 2)$ find the rate of change of T in the direction $\theta = \pi/3$.

We have
$$\frac{dT}{ds} = -\frac{64(2x)}{(x^2 + y^2 + 2)^2} \cos\theta - \frac{64(2y)}{(x^2 + y^2 + 2)^2} \sin\theta$$

At $(1, 2)$ in the direction $\theta = \dfrac{\pi}{3}$, $\dfrac{dT}{ds} = -\dfrac{128}{49}\dfrac{1}{2} - \dfrac{256}{49}\dfrac{\sqrt{3}}{2} = -\dfrac{64}{49}(1 + 2\sqrt{3})$.

5. The electrical potential V at any point (x, y) is given by $V = \ln\sqrt{x^2 + y^2}$. Find the rate of change of V at the point $(3, 4)$ in the direction toward the point $(2, 6)$.

Here,
$$\frac{dV}{ds} = \frac{x}{x^2 + y^2} \cos\theta + \frac{y}{x^2 + y^2} \sin\theta$$

Since θ is a second-quadrant angle and $\tan\theta = (6 - 4)/(2 - 3) = -2$, $\cos\theta = -1/\sqrt{5}$ and $\sin\theta = 2/\sqrt{5}$. Hence, at $(3, 4)$ in the indicated direction, $\dfrac{dV}{ds} = \dfrac{3}{25}\left(-\dfrac{1}{\sqrt{5}}\right) + \dfrac{4}{25}\dfrac{2}{\sqrt{5}} = \dfrac{\sqrt{5}}{25}$.

6. Find the maximum directional derivative for the surface and point of Problem 2.

At $P^*(7, 2)$ in the direction θ, $dz/ds = 14\cos\theta - 24\sin\theta$.

To find the value of θ for which $\dfrac{dz}{ds}$ is a maximum, set $\dfrac{d}{d\theta}\left(\dfrac{dz}{ds}\right) = -14\sin\theta - 24\cos\theta = 0$. Then $\tan\theta = -\frac{24}{14} = -\frac{12}{7}$ and θ is either a second- or fourth-quadrant angle. For the second-quadrant angle, $\sin\theta = 12/\sqrt{193}$ and $\cos\theta = -7/\sqrt{193}$. For the fourth-quadrant angle, $\sin\theta = -12/\sqrt{193}$ and $\cos\theta = 7/\sqrt{193}$.

Since $\dfrac{d^2}{d\theta^2}\left(\dfrac{dz}{ds}\right) = \dfrac{d}{d\theta}(-14\sin\theta - 24\cos\theta) = -14\cos\theta + 24\sin\theta$ is negative for the fourth-quadrant angle, the maximum directional derivative is $\dfrac{dz}{ds} = 14\left(\dfrac{7}{\sqrt{193}}\right) - 24\left(-\dfrac{12}{\sqrt{193}}\right) = 2\sqrt{193}$, and the direction is $\theta = 300°15'$.

7. Find the maximum directional derivative for the function and point of Problem 3.

At $P^*(0, 3)$ in the direction θ, $dz/ds = 3\cos\theta + \sin\theta$.

To find the value of θ for which $\dfrac{dz}{ds}$ is a maximum, set $\dfrac{d}{d\theta}\left(\dfrac{dz}{ds}\right) = -3\sin\theta + \cos\theta = 0$. Then $\tan\theta = \frac{1}{3}$ and θ is either a first- or third-quadrant angle.

Since $\dfrac{d^2}{d\theta^2}\left(\dfrac{dz}{ds}\right) = \dfrac{d}{d\theta}(-3\sin\theta + \cos\theta) = -3\cos\theta - \sin\theta$ is negative for the first-quadrant angle, the maximum directional derivative is $\dfrac{dz}{ds} = 3\dfrac{3}{\sqrt{10}} + \dfrac{1}{\sqrt{10}} = \sqrt{10}$, and the direction is $\theta = 18°26'$.

8. In Problem 5, show that V changes most rapidly along the set of radial lines through the origin.

At any point (x_1, y_1) in the direction θ, $\frac{dV}{ds} = \frac{x_1}{x_1^2 + y_1^2}\cos\theta + \frac{y_1}{x_1^2 + y_1^2}\sin\theta$. Now V changes most rapidly when $\frac{d}{d\theta}\left(\frac{dV}{ds}\right) = -\frac{x_1}{x_1^2 + y_1^2}\sin\theta + \frac{y_1}{x_1^2 + y_1^2}\cos\theta = 0$, and then $\tan\theta = \frac{y_1/(x_1^2 + y_1^2)}{x_1/(x_1^2 + y_1^2)} = \frac{y_1}{x_1}$. Thus, θ is the angle of inclination of the line joining the origin and the point (x_1, y_1).

9. Find the directional derivative of $F(x, y, z) = xy + 2xz - y^2 + z^2$ at the point $(1, -2, 1)$ along the curve $x = t$, $y = t - 3$, $z = t^2$ in the direction of increasing z.

A set of direction numbers of the tangent to the curve at $(1, -2, 1)$ is $[1, 1, 2]$; the direction cosines are $[1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}]$. The directional derivative is

$$\frac{\partial F}{\partial x}\cos\alpha + \frac{\partial F}{\partial y}\cos\beta + \frac{\partial F}{\partial z}\cos\gamma = 0\,\frac{1}{\sqrt{6}} + 5\,\frac{1}{\sqrt{6}} + 4\,\frac{2}{\sqrt{6}} = \frac{13\sqrt{6}}{6}$$

10. Examine $f(x, y) = x^2 + y^2 - 4x + 6y + 25$ for maximum and minimum values.

The conditions $\partial f/\partial x = 2x - 4 = 0$ and $\partial f/\partial y = 2y + 6 = 0$ are satisfied when $x = 2$, $y = -3$.

Since $f(x, y) = (x^2 - 4x + 4) + (y^2 + 6y + 9) + 25 - 4 - 9 = (x - 2)^2 + (y + 3)^2 + 12$, it is evident that $f(2, -3) = 12$ is a minimum value of the function.

Geometrically, $(2, -3, 12)$ is the minimum point of the surface $z = x^2 + y^2 - 4x + 6y + 25$.

11. Examine $f(x, y) = x^3 + y^3 + 3xy$ for maximum and minimum values.

The conditions $\partial f/\partial x = 3(x^2 + y) = 0$ and $\partial f/\partial y = 3(y^2 + x) = 0$ are satisfied when $x = 0$, $y = 0$ and when $x = -1$, $y = -1$.

At $(0, 0)$, $\frac{\partial^2 f}{\partial x^2} = 6x = 0$, $\frac{\partial^2 f}{\partial x\,\partial y} = 3$, and $\frac{\partial^2 f}{\partial y^2} = 6y = 0$. Then $\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = 9 > 0$, and $(0, 0)$ yields neither a maximum nor minimum.

At $(-1, -1)$, $\frac{\partial^2 f}{\partial x^2} = -6$, $\frac{\partial^2 f}{\partial x\,\partial y} = 3$, and $\frac{\partial^2 f}{\partial y^2} = -6$. Then $\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = -27 < 0$, and $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} < 0$. Hence, $f(-1, -1) = 1$ is the maximum value of the function.

12. Divide 120 into three parts such that the sum of their products taken two at a time is a maximum.

Let x, y, and $120 - (x + y)$ be the three parts. The function to be maximized is $S = xy + (x + y)(120 - x - y)$, and

$$\frac{\partial S}{\partial x} = y + (120 - x - y) - (x + y) = 120 - 2x - y \qquad \frac{\partial S}{\partial y} = x + (120 - x - y) - (x + y) = 120 - x - 2y$$

Setting $\frac{\partial S}{\partial x} = \frac{\partial S}{\partial y} = 0$ yields $2x + y = 120$ and $x + 2y = 120$. Simultaneous solution gives $x = 40$, $y = 40$, and $120 - (x + y) = 40$ as the three parts, and $S = 3(40^2) = 4800$. For $x = y = 1$, $S = 237$; hence, $S = 4800$ is the maximum value.

13. Find the point in the plane $2x - y + 2z = 16$ nearest the origin.

Let (x, y, z) be the required point; then the square of its distance from the origin is $D = x^2 + y^2 + z^2$. Since also $2x - y + 2z = 16$, we have $y = 2x + 2z - 16$ and $D = x^2 + (2x + 2z - 16)^2 + z^2$.

Then the conditions $\partial D/\partial x = 2x + 4(2x + 2z - 16) = 0$ and $\partial D/\partial z = 4(2x + 2z - 16) + 2z = 0$ are equivalent to $5x + 4z = 32$ and $4x + 5z = 32$, and $x = z = \frac{32}{9}$. Since it is known that a point for which D is a minimum exists, $(\frac{32}{9}, -\frac{16}{9}, \frac{32}{9})$ is that point.

14. Show that a rectangular parallelepiped of maximum volume V with constant surface area S is a cube.

Let the dimensions be x, y, and z. Then $V = xyz$ and $S = 2(xy + yz + zx)$.

The second relation may be solved for z and substituted in the first, to express V as a function of x and y. We prefer to avoid this step by simply treating z as a function of x and y. Then

$$\frac{\partial V}{\partial x} = yz + xy\frac{\partial z}{\partial x} \qquad \frac{\partial V}{\partial y} = xz + xy\frac{\partial z}{\partial y}$$

$$\frac{\partial S}{\partial x} = 0 = 2\left(y + z + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x}\right) \qquad \frac{\partial S}{\partial y} = 0 = 2\left(x + z + x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y}\right)$$

From the latter two equations, $\frac{\partial z}{\partial x} = -\frac{y+z}{x+y}$ and $\frac{\partial z}{\partial y} = -\frac{x+z}{x+y}$. Substituting in the first two yields the conditions $\frac{\partial V}{\partial x} = yz - \frac{xy(y+z)}{x+y} = 0$ and $\frac{\partial V}{\partial y} = xz - \frac{xy(x+z)}{x+y} = 0$, which reduce to $y^2(z-x) = 0$ and $x^2(z-y) = 0$. Thus $x = y = z$, as required.

15. Find the volume V of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let $P(x, y, z)$ be the vertex in the first octant. Then $V = 8xyz$. Consider z to be defined as a function of the independent variables x and y by the equation of the ellipsoid. The necessary conditions for a maximum are

$$\frac{\partial V}{\partial x} = 8\left(yz + xy\frac{\partial z}{\partial x}\right) = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = 8\left(xz + xy\frac{\partial z}{\partial y}\right) = 0 \tag{1}$$

From the equation of the ellipsoid, obtain $\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$ and $\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0$. Eliminate $\partial z/\partial x$ and $\partial z/\partial y$ between these relations and (1) to obtain

$$\frac{\partial V}{\partial x} = 8\left(yz - \frac{c^2x^2y}{a^2z}\right) = 0 \quad \text{and} \quad \frac{\partial V}{\partial y} = 8\left(xz - \frac{c^2xy^2}{b^2z}\right) = 0$$

and, finally,

$$\frac{x^2}{a^2} = \frac{z^2}{c^2} = \frac{y^2}{b^2} \tag{2}$$

Combine (2) with the equation of the ellipsoid to get $x = a\sqrt{3}/3$, $y = b\sqrt{3}/3$, and $z = c\sqrt{3}/3$. Then $V = 8xyz = (8\sqrt{3}/9)abc$ cubic units.

Supplementary Problems

16. Find the directional derivative of the given function at the given point in the indicated direction:
(a) $z = x^2 + xy + y^2$, $(3, 1)$, $\theta = \pi/3$ (b) $z = x^3 + y^3 - 3xy$, $(2, 1)$, $\theta = \arctan 2/3$
(c) $z = y + x\cos xy$, $(0, 0)$, $\theta = \pi/3$ (d) $z = 2x^2 + 3xy - y^2$, $(1, -1)$, toward $(2, 1)$

Ans. (a) $\frac{1}{2}(7 + 5\sqrt{3})$; (b) $21\sqrt{13}/13$; (c) $\frac{1}{2}(1 + \sqrt{3})$; (d) $11\sqrt{5}/5$

17. Find the maximum directional derivative for each of the functions of Problem 16 at the given point.

Ans. (a) $\sqrt{74}$; (b) $3\sqrt{10}$; (c) $\sqrt{2}$; (d) $\sqrt{26}$

18. Show that the maximum directional derivative of $V = \ln\sqrt{x^2 + y^2}$ of Problem 8 is constant along any circle $x^2 + y^2 = r^2$.

19. On a hill represented by $z = 8 - 4x^2 - 2y^2$, find (a) the direction of the steepest grade at $(1, 1, 2)$ and (b) the direction of the contour line (direction for which z = constant). Note that the directions are mutually perpendicular. *Ans.* (a) $\arctan\frac{1}{2}$, third quadrant; (b) $\arctan -2$

20. Show that the sum of the squares of the directional derivatives of $z = f(x, y)$ at any of its points is constant for any two mutually perpendicular directions and is equal to the square of the maximum directional derivative.

21. Given $z = f(x, y)$ and $w = g(x, y)$ such that $\partial z/\partial x = \partial w/\partial y$ and $\partial z/\partial y = -\partial w/\partial x$. If θ_1 and θ_2 are two mutually perpendicular directions, show that at any point $P(x, y)$, $\partial z/\partial s_1 = \partial w/\partial s_2$ and $\partial z/\partial s_2 = -\partial w/\partial s_1$.

22. Find the directional derivative of the given function at the given point in the indicated direction:
(*a*) xy^2z, $(2, 1, 3)$, $[1, -2, 2]$
(*b*) $x^2 + y^2 + z^2$, $(1, 1, 1)$, toward $(2, 3, 4)$
(*c*) $x^2 + y^2 - 2xz$, $(1, 3, 2)$, along $x^2 + y^2 - 2xz = 6$, $3x^2 - y^2 + 3z = 0$ in the direction of increasing z

Ans. (*a*) $-\frac{17}{3}$; (*b*) $6\sqrt{14}/7$; (*c*) 0

23. Examine each of the following functions for relative maximum and minimum values.
(*a*) $z = 2x + 4y - x^2 - y^2 - 3$ *Ans.* maximum $= 2$ when $x = 1$, $y = 2$
(*b*) $z = x^3 + y^3 - 3xy$ *Ans.* minimum $= -1$ when $x = 1$, $y = 1$
(*c*) $z = x^2 + 2xy + 2y^2$ *Ans.* minimum $= 0$ when $x = 0$, $y = 0$
(*d*) $z = (x - y)(1 - xy)$ *Ans.* neither maximum nor minimum
(*e*) $z = 2x^2 + y^2 + 6xy + 10x - 6y + 5$ *Ans.* neither maximum nor minimum
(*f*) $z = 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3$ *Ans.* minimum $= -\sqrt{6}$ when $x = -\sqrt{6}/6$, $y = \sqrt{6}/3$; maximum $= \sqrt{6}$ when $x = \sqrt{6}/6$, $y = -\sqrt{6}/3$
(*g*) $z = xy(2x + 4y + 1)$ *Ans.* maximum $= \frac{1}{216}$ when $x = -\frac{1}{6}$, $y = -\frac{1}{12}$

24. Find positive numbers x, y, z such that
(*a*) $x + y + z = 18$ and xyz is a maximum
(*b*) $xyz = 27$ and $x + y + z$ is a minimum
(*c*) $x + y + z = 20$ and xyz^2 is a maximum
(*d*) $x + y + z = 12$ and xy^2z^3 is a maximum

Ans. (*a*) $x = y = z = 6$; (*b*) $x = y = z = 3$; (*c*) $x = y = 5$, $z = 10$; (*d*) $x = 2$, $y = 4$, $z = 6$

25. Find the minimum value of the square of the distance from the origin to the plane $Ax + By + Cz + D = 0$. *Ans.* $D^2/(A^2 + B^2 + C^2)$

26. (*a*) The surface area of a rectangular box without a top is to be 108 ft^2. Find the greatest possible volume. (*b*) The volume of a rectangular box without a top is to be 500 ft^3. Find the minimum surface area. *Ans.* (*a*) 108 ft^3; (*b*) 300 ft^2

27. Find the point on $z = xy - 1$ nearest the origin. *Ans.* $(0, 0, -1)$

28. Find the equation of the plane through $(1, 1, 2)$ that cuts off the least volume in the first octant.

Ans. $2x + 2y + z = 6$

29. Determine the values of p and q so that the sum S of the squares of the vertical distances of the points $(0, 2)$, $(1, 3)$, and $(2, 5)$ from the line $y = px + q$ is a minimum. (*Hint*: $S = (q - 2)^2 + (p + q - 3)^2 + (2p + q - 5)^2$.) *Ans.* $p = \frac{3}{2}$; $q = \frac{11}{6}$

Chapter 68

Vector Differentiation and Integration

VECTOR DIFFERENTIATION. Let

$$\mathbf{r} = \mathbf{i}f_1(t) + \mathbf{j}f_2(t) + \mathbf{k}f_3(t) = \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3$$
$$\mathbf{s} = \mathbf{i}g_1(t) + \mathbf{j}g_2(t) + \mathbf{k}g_3(t) = \mathbf{i}g_1 + \mathbf{j}g_2 + \mathbf{k}g_3$$
$$\mathbf{u} = \mathbf{i}h_1(t) + \mathbf{j}h_2(t) + \mathbf{k}h_3(t) = \mathbf{i}h_1 + \mathbf{j}h_2 + \mathbf{k}h_3$$

be vectors whose components are functions of a single scalar variable t having continuous first and second derivatives.

We can show, as in Chapter 23 for plane vectors, that

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) = \frac{d\mathbf{r}}{dt}\cdot s + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt} \tag{68.1}$$

Also, from the properties of determinants whose entries are functions of a single variable, we have

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{s}) = \frac{d}{dt}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \end{vmatrix} \tag{68.2}$$
$$= \frac{d\mathbf{r}}{dt}\times\mathbf{s} + \mathbf{r}\times\frac{d\mathbf{s}}{dt}$$

and

$$\frac{d}{dt}[\mathbf{r}\cdot(\mathbf{s}\times\mathbf{u})] = \frac{d\mathbf{r}}{dt}\cdot(\mathbf{s}\times\mathbf{u}) + \mathbf{r}\cdot\left(\frac{d\mathbf{s}}{dt}\times\mathbf{u}\right) + \mathbf{r}\cdot\left(\mathbf{s}\times\frac{d\mathbf{u}}{dt}\right) \tag{68.3}$$

These formulas may also be established by expanding the products before differentiating.

From (*68.2*) follows

$$\frac{d}{dt}[\mathbf{r}\times(\mathbf{s}\times\mathbf{u})] = \frac{d\mathbf{r}}{dt}\times(\mathbf{s}\times\mathbf{u}) + \mathbf{r}\times\frac{d}{dt}(\mathbf{s}\times\mathbf{u})$$
$$= \frac{d\mathbf{r}}{dt}\times(\mathbf{s}\times\mathbf{u}) + \mathbf{r}\times\left(\frac{d\mathbf{s}}{dt}\times\mathbf{u}\right) + \mathbf{r}\times\left(\mathbf{s}\times\frac{d\mathbf{u}}{dt}\right) \tag{68.4}$$

SPACE CURVES. Consider the space curve

$$x = f(t) \qquad y = g(t) \qquad z = h(t) \tag{68.5}$$

where $f(t)$, $g(t)$, and $h(t)$ have continuous first and second derivatives. Let the position vector of a general variable point $P(x, y, z)$ of the curve be given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

As in Chapter 23, $\mathbf{t} = d\mathbf{r}/ds$ is the unit tangent vector to the curve. If $\mathbf{R}$ is the position vector of a point (X, Y, Z) on the tangent line at P, the vector equation of this line is (see Chapter 65)

$$\mathbf{R} - \mathbf{r} = k\mathbf{t} \qquad \text{for } k \text{ a scalar variable} \tag{68.6}$$

and the equations in rectangular coordinates are

$$\frac{X-x}{dx/ds} = \frac{Y-y}{dy/ds} = \frac{Z-z}{dz/ds}$$

where $\left[\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right]$ is a set of direction cosines of the line. In the corresponding equation, (*66.2*), a set of direction numbers $\left[\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right]$ was used.

The vector equation of the normal plane to the curve at P is given by

$$(\mathbf{R}-\mathbf{r})\cdot\mathbf{t}=0 \tag{68.7}$$

where $\mathbf{R}$ is the position vector of a general point of the plane.

Again, as in Chapter 23, $d\mathbf{t}/ds$ is a vector perpendicular to $\mathbf{t}$. If $\mathbf{n}$ is a unit vector having the direction of $d\mathbf{t}/ds$, then

$$\frac{d\mathbf{t}}{ds}=|K|\mathbf{n}$$

where $|K|$ is the magnitude of the curvature at P. The unit vector

$$\mathbf{n}=\frac{1}{|K|}\frac{d\mathbf{t}}{ds} \tag{68.8}$$

is called the *principal normal* to the curve at P.

The unit vector $\mathbf{b}$ at P, defined by

$$\mathbf{b}=\mathbf{t}\times\mathbf{n} \tag{68.9}$$

is called the *binormal* at P. The three vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ form at P a right-handed triad of mutually orthogonal vectors. (See Problems 1 and 2.)

At a general point P of a space curve (Fig. 68-1), the vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ determine three mutually perpendicular planes:

1. The *osculating plane*, containing $\mathbf{t}$ and $\mathbf{n}$, of equation $(\mathbf{R}-\mathbf{r})\cdot\mathbf{b}=0$
2. The *normal plane*, containing $\mathbf{n}$ and $\mathbf{b}$, of equation $(\mathbf{R}-\mathbf{r})\cdot\mathbf{t}=0$
3. The *rectifying plane*, containing $\mathbf{t}$ and $\mathbf{b}$, of equation $(\mathbf{R}-\mathbf{r})\cdot\mathbf{n}=0$

In each equation, $\mathbf{R}$ is the position vector of a general point in the particular plane.

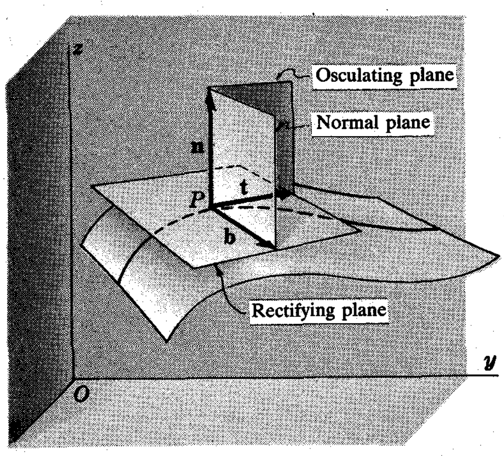

Fig. 68-1

SURFACES. Let $F(x, y, z)=0$ be the equation of a surface. (See Chapter 66.) A parametric representation results when x, y, and z are written as functions of two independent variables or parameters u and v, for example, as

$$x=f_1(u,v) \qquad y=f_2(u,v) \qquad z=f_3(u,v) \tag{68.10}$$

When u is replaced with u_0, a constant, (*68.10*) becomes

$$x=f_1(u_0,v) \qquad y=f_2(u_0,v) \qquad z=f_3(u_0,v) \tag{68.11}$$

the equation of a space curve (u curve) lying on the surface. Similarly, when v is replaced with v_0, a constant, (*68.10*) becomes

$$x = f_1(u, v_0) \qquad y = f_2(u, v_0) \qquad z = f_3(u, v_0) \tag{68.12}$$

the equation of another space curve (v curve) on the surface. The two curves intersect in a point of the surface obtained by setting $u = u_0$ and $v = v_0$ simultaneously in (*68.10*).

The position vector of a general point P on the surface is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{i}f_1(u, v) + \mathbf{j}f_2(u, v) + \mathbf{k}f_3(u, v) \tag{68.13}$$

Suppose (*68.11*) and (*68.12*) are the u and v curves through P. Then, at P,

$$\frac{\partial \mathbf{r}}{\partial v} = \mathbf{i}\frac{\partial}{\partial v} f_1(u_0, v) + \mathbf{j}\frac{\partial}{\partial v} f_2(u_0, v) + \mathbf{k}\frac{\partial}{\partial v} f_3(u_0, v)$$

is a vector tangent to the u curve, and

$$\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i}\frac{\partial}{\partial u} f_1(u, v_0) + \mathbf{j}\frac{\partial}{\partial u} f_2(u, v_0) + \mathbf{k}\frac{\partial}{\partial u} f_3(u, v_0)$$

is a vector tangent to the v curve. The two tangents determine a plane that is the tangent plane to the surface at P (Fig. 68-2). Clearly, a normal to this plane is given by $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$. The *unit normal* to the surface at P is defined by

$$\mathbf{n} = \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left|\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}\right|} \tag{68.14}$$

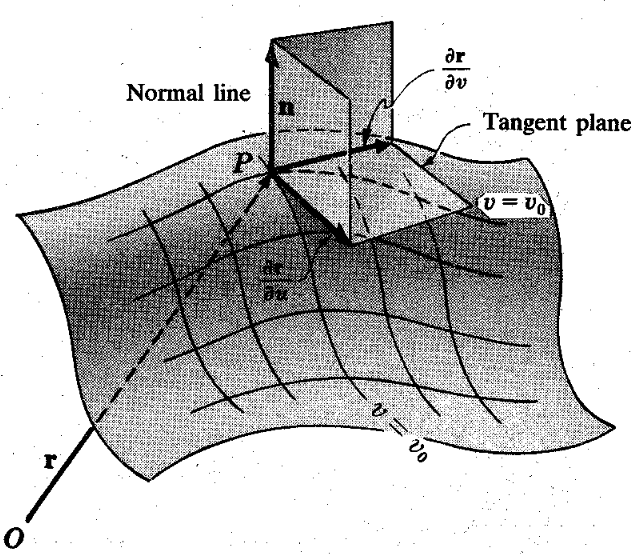

Fig. 68-2

If $\mathbf{R}$ is the position vector of a general point on the normal to the surface at P, its vector equation is

$$(\mathbf{R} - \mathbf{r}) = k\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \tag{68.15}$$

If $\mathbf{R}$ is the position vector of a general point on the tangent plane to the surface at P, its vector equation is

$$(\mathbf{R} - \mathbf{r}) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) = 0 \tag{68.16}$$

(See Problem 3.)

THE OPERATOR ∇. In Chapter 67 the directional derivative of $z = f(x, y)$ at an arbitrary point (x, y) and in a direction making an angle θ with the positive x axis is given as

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

Let us write

$$\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y}\right)\cdot(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) \tag{68.17}$$

Now $\mathbf{a} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ is a unit vector whose direction makes the angle θ with the positive x axis. The other factor on the right of (*68.17*), when written as $\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right)f$, suggests the definition of a vector differential operator ∇ (del), defined by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \tag{68.18}$$

In vector analysis, $\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y}$ is called the *gradient* of f or *grad f*. From (*68.17*), we see that the component of ∇f in the direction of a *unit vector* $\mathbf{a}$ is the directional derivative of f in the direction of $\mathbf{a}$.

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector to $P(x, y)$. Since

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y}\right)\cdot\left(\mathbf{i}\frac{dx}{ds} + \mathbf{j}\frac{dy}{ds}\right)$$

$$= \nabla f\cdot\frac{d\mathbf{r}}{ds}$$

and

$$\left|\frac{df}{ds}\right| = |\nabla f|\cos\phi$$

where ϕ is the angle between the vectors ∇f and $d\mathbf{r}/ds$, we see that df/ds is maximal when $\cos\phi = 1$, that is, when ∇f and $d\mathbf{r}/ds$ have the same direction. Thus, the maximum value of the directional derivative at P is $|\nabla f|$; and its direction is that of ∇f. (Compare the discussion of maximum directional derivatives in Chapter 67.) (See Problem 4.)

For $w = F(x, y, z)$, we define

$$\nabla F = \mathbf{i}\frac{\partial F}{\partial x} + \mathbf{j}\frac{\partial F}{\partial y} + \mathbf{k}\frac{\partial F}{\partial z}$$

and the directional derivative of $F(x, y, z)$ at an arbitrary point $P(x, y, z)$ in the direction $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_2\mathbf{k}$ is

$$\frac{dF}{ds} = \nabla F\cdot\mathbf{a} \tag{68.19}$$

As in the case of functions of two variables, $|\nabla F|$ is the maximum value of the directional derivative of $F(x, y, z)$ at $P(x, y, z)$, and its direction is that of ∇F. (See Problem 5.)

Consider now the surface $F(x, y, z) = 0$. The equation of the tangent plane to the surface at one of its points $P_0(x_0, y_0, z_0)$ is given by

$$(x - x_0)\frac{\partial F}{\partial x} + (y - y_0)\frac{\partial F}{\partial y} + (z - z_0)\frac{\partial F}{\partial z}$$

$$= [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}]\cdot\left[\mathbf{i}\frac{\partial F}{\partial x} + \mathbf{j}\frac{\partial F}{\partial y} + \mathbf{k}\frac{\partial F}{\partial z}\right] = 0 \tag{68.20}$$

with the understanding that the partial derivatives are evaluated at P_0. The first factor is an arbitrary vector through P_0 in the tangent plane; hence the second factor ∇F, evaluated at P_0, is normal to the tangent plane, that is, is normal to the surface at P_0. (See Problems 6 and 7.)

DIVERGENCE AND CURL. The *divergence* of a vector $\mathbf{F} = \mathbf{i}f_1(x, y, z) + \mathbf{j}f_2(x, y, z) + \mathbf{k}f_3(x, y, z)$, sometimes called *del dot* **F**, is defined by

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3 \qquad (68.21)$$

The *curl* of the vector **F**, or *del cross* **F**, is defined by

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} f_3 - \frac{\partial}{\partial z} f_2\right)\mathbf{i} + \left(\frac{\partial}{\partial z} f_1 - \frac{\partial}{\partial x} f_3\right)\mathbf{j} + \left(\frac{\partial}{\partial x} f_2 - \frac{\partial}{\partial y} f_1\right)\mathbf{k} \qquad (68.22)$$

(See Problem 8.)

INTEGRATION. Our discussion of integration here will be limited to ordinary integration of vectors and to so-called "line integrals." As an example of the former, let

$$\mathbf{F}(u) = \mathbf{i}\cos u + \mathbf{j}\sin u + au\mathbf{k}$$

be a vector depending upon the scalar variable u. Then

$$\mathbf{F}'(u) = -\mathbf{i}\sin u + \mathbf{j}\cos u + a\mathbf{k}$$

and

$$\int \mathbf{F}'(u)\,du = \int (-\mathbf{i}\sin u + \mathbf{j}\cos u + a\mathbf{k})\,du$$

$$= \mathbf{i}\int -\sin u\,du + \mathbf{j}\int \cos u\,du + \mathbf{k}\int a\,du$$

$$= \mathbf{i}\cos u + \mathbf{j}\sin u + au\mathbf{k} + \mathbf{c}$$

$$= \mathbf{F}(u) + \mathbf{c}$$

where **c** is an arbitrary constant vector independent of u. Moreover,

$$\int_{u=a}^{u=b} \mathbf{F}'(u)\,du = [\mathbf{F}(u) + \mathbf{c}]_{u=a}^{u=b} = \mathbf{F}(b) - \mathbf{F}(a)$$

(See Problems 9 and 10.)

LINE INTEGRALS. Consider two points P_0 and P_1 in space, joined by an arc C. The arc may be the segment of a straight line or a portion of a space curve $x = g_1(t)$, $y = g_2(t)$, $z = g_3(t)$, or it may consist of several subarcs of curves. In any case, C is assumed to be continuous at each of its points and not to intersect itself. Consider further a vector function

$$\mathbf{F} = \mathbf{F}(x, y, z) = \mathbf{i}f_1(x, y, z) + \mathbf{j}f_2(x, y, z) + \mathbf{k}f_3(x, y, z)$$

which at every point in a region about C, and, in particular, at every point of C, defines a vector of known magnitude and direction. Denote by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad (68.23)$$

the position vector of $P(x, y, z)$ on C. The integral

$$\underset{C}{\int_{P_0}^{P_1}} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right) ds = \underset{C}{\int_{P_0}^{P_1}} \mathbf{F} \cdot d\mathbf{r} \qquad (68.24)$$

is called a line integral, that is, an integral along a given path C.

As an example, let $\mathbf{F}$ denote a force. The work done by it in moving a particle over $d\mathbf{r}$ is given by (see Problem 9 of Chapter 23)

$$|\mathbf{F}||d\mathbf{r}|\cos\theta = \mathbf{F}\cdot d\mathbf{r}$$

and the work done in moving the particle from P_0 to P_1 along the arc C is given by

$$\int_{P_0\,C}^{P_1} \mathbf{F}\cdot d\mathbf{r}$$

From (*68.23*),

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$$

and (*68.24*) becomes

$$\int_{P_0\,C}^{P_1} \mathbf{F}\cdot d\mathbf{r} = \int_{P_0\,C}^{P_1} (f_1\,dx + f_2\,dy + f_3\,dz) \tag{68.25}$$

(See Problem 11.)

Solved Problems

1. A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the magnitude of its velocity and acceleration at times $t = 0$ and $t = \frac{1}{2}\pi$.

Let $P(x, y, z)$ be a point on the curve, and

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 4\mathbf{i}\cos t + 4\mathbf{j}\sin t + 6\mathbf{k}t$$

be its position vector. Then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -4\mathbf{i}\sin t + 4\mathbf{j}\cos t + 6\mathbf{k} \qquad \text{and} \qquad \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -4\mathbf{i}\cos t - 4\mathbf{j}\sin t$$

At $t = 0$: $\quad \mathbf{v} = 4\mathbf{j} + 6\mathbf{k} \quad |\mathbf{v}| = \sqrt{16+36} = 2\sqrt{13}$

$\quad \mathbf{a} = -4\mathbf{i} \quad |\mathbf{a}| = 4$

At $t = \frac{1}{2}\pi$: $\quad \mathbf{v} = -4\mathbf{i} + 6\mathbf{k} \quad |\mathbf{v}| = \sqrt{16+36} = 2\sqrt{13}$

$\quad \mathbf{a} = -4\mathbf{j} \quad |\mathbf{a}| = 4$

2. At the point $(1, 1, 1)$ or $t = 1$ of the space curve $x = t$, $y = t^2$, $z = t^3$, find
(*a*) The equations of the tangent line and normal plane
(*b*) The unit tangent, principal normal, and binormal
(*c*) The equations of the principal normal and binormal

We have

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\,\frac{dt}{ds} = \frac{\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}}{\sqrt{1 + 4t^2 + 9t^4}}$$

At $t = 1$, $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{t} = \dfrac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.

(*a*) If $\mathbf{R}$ is the position vector of a general point (X, Y, Z) on the tangent line, its vector equation is $\mathbf{R} - \mathbf{r} = k\mathbf{t}$ or

$$(X-1)\mathbf{i} + (Y-1)\mathbf{j} + (Z-1)\mathbf{k} = \frac{k}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

and its rectangular equations are

$$\frac{X-1}{1} = \frac{Y-1}{2} = \frac{Z-1}{3}$$

If $\mathbf{R}$ is the position vector of a general point (X, Y, Z) on the normal plane, its vector equation is $(\mathbf{R} - \mathbf{r}) \cdot \mathbf{t} = 0$ or

$$[(X-1)\mathbf{i} + (Y-1)\mathbf{j} + (Z-1)\mathbf{k}] \cdot \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 0$$

and its rectangular equation is

$$(X-1) + 2(Y-1) + 3(Z-1) = X + 2Y + 3Z - 6 = 0$$

(see Problem 2(*a*) of Chapter 66.)

(*b*)
$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt}\frac{dt}{ds} = \frac{(-4t - 18t^3)\mathbf{i} + (2 - 18t^4)\mathbf{j} + (6t + 12t^3)\mathbf{k}}{(1 + 4t^2 + 9t^4)^2}$$

At $t = 1$, $\dfrac{d\mathbf{t}}{ds} = \dfrac{-11\mathbf{i} - 8\mathbf{j} + 9\mathbf{k}}{98}$ and $\left|\dfrac{d\mathbf{t}}{ds}\right| = \dfrac{1}{7}\sqrt{\dfrac{19}{14}} = |K|$. Then

$$\mathbf{n} = \frac{1}{|K|}\frac{d\mathbf{t}}{ds} = \frac{-11\mathbf{i} - 8\mathbf{j} + 9\mathbf{k}}{\sqrt{266}}$$

and
$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \frac{1}{\sqrt{14}\sqrt{266}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \frac{1}{\sqrt{19}}(3\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

(*c*) If $\mathbf{R}$ is the position vector of a general point (X, Y, Z) on the principal normal, its vector equation is $\mathbf{R} - \mathbf{r} = k\mathbf{n}$ or

$$(X-1)\mathbf{i} + (Y-1)\mathbf{j} + (Z-1)\mathbf{k} = k\,\frac{-11\mathbf{i} - 8\mathbf{j} + 9\mathbf{k}}{\sqrt{266}}$$

and the equations in rectangular coordinates are

$$\frac{X-1}{-11} = \frac{Y-1}{-8} = \frac{Z-1}{9}$$

If $\mathbf{R}$ is the position vector of a general point (X, Y, Z) on the binormal, its vector equation is $\mathbf{R} - \mathbf{r} = k \cdot \mathbf{b}$ or

$$(X-1)\mathbf{i} + (Y-1)\mathbf{j} + (Z-1)\mathbf{k} = k\,\frac{3\mathbf{i} - 3\mathbf{j} + \mathbf{k}}{\sqrt{19}}$$

and the equations in rectangular coordinates are

$$\frac{X-1}{3} = \frac{Y-1}{-3} = \frac{Z-1}{1}$$

3. Find the equations of the tangent plane and normal line to the surface $x = 2(u + v)$, $y = 3(u - v)$, $z = uv$ at the point $P(u = 2, v = 1)$.

Here $\quad \mathbf{r} = 2(u+v)\mathbf{i} + 3(u-v)\mathbf{j} + uv\mathbf{k} \qquad \dfrac{\partial \mathbf{r}}{\partial u} = 2\mathbf{i} + 3\mathbf{j} + v\mathbf{k} \qquad \dfrac{\partial \mathbf{r}}{\partial v} = 2\mathbf{i} - 3\mathbf{j} + u\mathbf{k}$

and at the point P,

$$\mathbf{r} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \qquad \frac{\partial \mathbf{r}}{\partial u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \qquad \frac{\partial \mathbf{r}}{\partial v} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

and $$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = 9\mathbf{i} - 2\mathbf{j} - 12\mathbf{k}$$

The vector and rectangular equations of the normal line are

$$\mathbf{R} - \mathbf{r} = k\,\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

or $$(X-6)\mathbf{i} + (Y-3)\mathbf{j} + (Z-2)\mathbf{k} = k(9\mathbf{i} - 2\mathbf{j} - 12\mathbf{k})$$

and $$\frac{X-6}{9} + \frac{Y-3}{-2} = \frac{Z-2}{-12}$$

The vector and rectangular equations of the tangent plane are

$$(\mathbf{R} - \mathbf{r}) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) = 0$$

or $$[(X-6)\mathbf{i} + (Y-3)\mathbf{j} + (Z-2)\mathbf{k}] \cdot [9\mathbf{i} - 2\mathbf{j} - 12\mathbf{k}] = 0$$

and $$9X - 2Y - 12Z - 24 = 0$$

4. (*a*) Find the directional derivative of $f(x, y) = x^2 - 6y^2$ at the point $(7, 2)$ in the direction $\theta = \frac{1}{4}\pi$.
(*b*) Find the maximum value of the directional derivative at $(7, 2)$.

(*a*) $$\nabla f = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y}\right)(x^2 - 6y^2) = \mathbf{i}\,\frac{\partial}{\partial x}(x^2 - 6y^2) + \mathbf{j}\,\frac{\partial}{\partial y}(x^2 - 6y^2) = 2x\mathbf{i} - 12y\mathbf{j}$$

and $$\mathbf{a} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta = \frac{1}{\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j}$$

At $(7, 2)$, $\nabla f = 14\mathbf{i} - 24\mathbf{j}$, and

$$\nabla f \cdot \mathbf{a} = (14\mathbf{i} - 24\mathbf{j}) \cdot \left(\frac{1}{\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j}\right) = 7\sqrt{2} - 12\sqrt{2} = -5\sqrt{2}$$

is the directional derivative.

(*b*) At $(7, 2)$, with $\nabla f = 14\mathbf{i} - 24\mathbf{j}$, $|\nabla f| = \sqrt{14^2 + 24^2} = 2\sqrt{193}$ is the maximum directional derivative. Since

$$\frac{\nabla f}{|\nabla f|} = \frac{7}{\sqrt{193}}\,\mathbf{i} - \frac{12}{\sqrt{193}}\,\mathbf{j} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$$

the direction is $\theta = 300°15'$. (See Problems 2 and 6 of Chapter 67.)

5. (*a*) Find the directional derivative of $F(x, y, z) = x^2 - 2y^2 + 4z^2$ at $P(1, 1, -1)$ in the direction $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.
(*b*) Find the maximum value of the directional derivative at P.

Here $$\nabla F = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)(x^2 - 2y^2 + 4z^2) = 2x\mathbf{i} - 4y\mathbf{j} + 8z\mathbf{k}$$

and at $(1, 1, -1)$, $\nabla F = 2\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$.
(*a*) $\nabla F \cdot \mathbf{a} = (2\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = 8$
(*b*) At P, $|\nabla F| = \sqrt{84} = 2\sqrt{21}$. The direction is $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$.

6. Given the surface $F(x, y, z) = x^3 + 3xyz + 2y^3 - z^3 - 5 = 0$ and one of its points $P_0(1, 1, 1)$, find (*a*) a unit normal to the surface at P_0, (*b*) the equations of the normal line at P_0, and (*c*) the equation of the tangent plane at P_0.

Here $$\nabla F = (3x^2 + 3yz)\mathbf{i} + (3xz + 6y^2)\mathbf{j} + (3xy - 3z^2)\mathbf{k}$$

and at $P_0(1, 1, 1)$, $\nabla F = 6\mathbf{i} + 9\mathbf{j}$.

(*a*) $\dfrac{\nabla F}{|\nabla F|} = \dfrac{2}{\sqrt{13}}\,\mathbf{i} + \dfrac{3}{\sqrt{13}}\,\mathbf{j}$ is a unit normal at P_0; the other is $-\dfrac{2}{\sqrt{13}}\,\mathbf{i} - \dfrac{2}{\sqrt{13}}\,\mathbf{j}$.

(*b*) The equations of the normal line are $\dfrac{X-1}{2} = \dfrac{Y-1}{3}$, $Z = 1$.

(*c*) The equation of the tangent plane is $2(X-1) + 3(Y-1) = 2X + 3Y - 5 = 0$.

7. Find the angle of intersection of the surfaces

$$F_1 = x^2 + y^2 + z^2 - 9 = 0 \qquad \text{and} \qquad F_2 = x^2 + 2y^2 - z - 8 = 0$$

at the point $(2, 1, -2)$.

We have
$$\nabla F_1 = \nabla(x^2 + y^2 + z^2 - 9) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$
and
$$\nabla F_2 = \nabla(x^2 + 2y^2 - z - 8) = 2x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$$

At $(2, 1, -2)$, $\nabla F_1 = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ and $\nabla F_2 = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

Now $\nabla F_1 \cdot \nabla F_2 = |\nabla F_1||\nabla F_2| \cos\theta$, where θ is the required angle. Thus,

$$(4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) \cdot (4\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = |4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}||4\mathbf{i} + 4\mathbf{j} - \mathbf{k}| \cos\theta$$

from which $\cos\theta = \frac{14}{99}\sqrt{33} = 0.81236$, and $\theta = 35°40'$.

8. When $\mathbf{B} = xy^2\mathbf{i} + 2x^2yz\mathbf{j} - 3yz^2\mathbf{k}$, find (*a*) div **B** and (*b*) curl **B**.

(*a*)
$$\begin{aligned}\operatorname{div}\mathbf{B} = \nabla\cdot\mathbf{B} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(xy^2\mathbf{i} + 2x^2yz\mathbf{j} - 3yz^2\mathbf{k})\\ &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)\\ &= y^2 + 2x^2z - 6yz\end{aligned}$$

(*b*)
$$\operatorname{curl}\mathbf{B} = \nabla\times\mathbf{B} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ xy^2 & 2x^2yz & -3yz^2\end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]\mathbf{k}$$

$$= -(3z^2 + 2x^2y)\mathbf{i} + (4xyz - 2xy)\mathbf{k}$$

9. Given $\mathbf{F}(u) = u\mathbf{i} + (u^2 - 2u)\mathbf{j} + (3u^2 + u^3)\mathbf{k}$, find (*a*) $\int \mathbf{F}(u)\,du$ and (*b*) $\int_0^1 \mathbf{F}(u)\,du$.

(*a*)
$$\begin{aligned}\int \mathbf{F}(u)\,du &= \int [u\mathbf{i} + (u^2 - 2u)\mathbf{j} + (3u^2 + u^3)\mathbf{k}]\,du\\ &= \mathbf{i}\int u\,du + \mathbf{j}\int (u^2 - 2u)\,du + \mathbf{k}\int (3u^2 + u^3)\,du\\ &= \frac{u^2}{2}\mathbf{i} + \left(\frac{u^3}{3} - u^2\right)\mathbf{j} + \left(u^3 + \frac{u^4}{4}\right)\mathbf{k} + \mathbf{c}\end{aligned}$$

where $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ with c_1, c_2, c_3 arbitrary scalars.

(*b*)
$$\int_0^1 \mathbf{F}(u)\,du = \left[\frac{u^2}{2}\mathbf{i} + \left(\frac{u^3}{3} - u^2\right)\mathbf{j} + \left(u^3 + \frac{u^4}{4}\right)\mathbf{k}\right]_0^1 = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{5}{4}\mathbf{k}$$

10. The acceleration of a particle at any time $t \geq 0$ is given by $\mathbf{a} = d\mathbf{v}/dt = e^t\mathbf{i} + e^{2t}\mathbf{j} + \mathbf{k}$. If at $t = 0$, the displacement is $\mathbf{r} = 0$ and the velocity is $\mathbf{v} = \mathbf{i} + \mathbf{j}$, find $\mathbf{r}$ and $\mathbf{v}$ at any time t.

Here
$$\mathbf{v} = \int \mathbf{a}\, dt = \mathbf{i}\int e^t\, dt + \mathbf{j}\int e^{2t}\, dt + \mathbf{k}\int dt$$
$$= e^t\mathbf{i} + \tfrac{1}{2}e^{2t}\mathbf{j} + t\mathbf{k} + \mathbf{c}_1$$

At $t = 0$, we have $\mathbf{v} = \mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{c}_1 = \mathbf{i} + \mathbf{j}$, from which $\mathbf{c}_1 = \frac{1}{2}\mathbf{j}$. Then
$$\mathbf{v} = e^t\mathbf{i} + \tfrac{1}{2}(e^{2t} + 1)\mathbf{j} + t\mathbf{k}$$
and
$$\mathbf{r} = \int \mathbf{v}\, dt = e^t\mathbf{i} + (\tfrac{1}{4}e^{2t} + \tfrac{1}{2}t)\mathbf{j} + \tfrac{1}{2}t^2\mathbf{k} + \mathbf{c}_2$$

At $t = 0$, $\mathbf{r} = \mathbf{i} + \frac{1}{4}\mathbf{j} + \mathbf{c}_2 = 0$, from which $\mathbf{c}_2 = -\mathbf{i} - \frac{1}{4}\mathbf{j}$. Thus,
$$\mathbf{r} = (e^t - 1)\mathbf{i} + (\tfrac{1}{4}e^{2t} + \tfrac{1}{2}t - \tfrac{1}{4})\mathbf{j} + \tfrac{1}{2}t^2\mathbf{k}$$

11. Find the work done by a force $\mathbf{F} = (x + yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + xy)\mathbf{k}$ in moving a particle from the origin O to $C(1, 1, 1)$, (*a*) along the straight line OC; (*b*) along the curve $x = t$, $y = t^2$, $z = t^3$; and (*c*) along the straight lines from O to $A(1, 0, 0)$, A to $B(1, 1, 0)$, and B to C.

$$\mathbf{F}\cdot d\mathbf{r} = [(x + yz)\mathbf{i} + (y + xz)\mathbf{j} + (z + xy)\mathbf{k}]\cdot[\mathbf{i}\, dx + \mathbf{j}\, dy + \mathbf{k}\, dz]$$
$$= (x + yz)\, dx + (y + xz)\, dy + (z + xy)\, dz$$

(*a*) Along the line OC, $x = y = z$ and $dx = dy = dz$. The integral to be evaluated becomes
$$W = \int_{\substack{(0,0,0)\\ C}}^{(1,1,1)} \mathbf{F}\cdot d\mathbf{r} = 3\int_0^1 (x + x^2)\, dx = [(\tfrac{3}{2}x^2 + x^3)]_0^1 = \tfrac{5}{2}$$

(*b*) Along the given curve, $x = t$ and $dx = dt$; $y = t^2$ and $dy = 2t\, dt$; $z = t^3$ and $dz = 3t^2\, dt$. At O, $t = 0$; at C, $t = 1$. Then
$$W = \int_0^1 (t + t^5)\, dt + (t^2 + t^4)2t\, dt + (t^3 + t^3)3t^2\, dt$$
$$= \int_0^1 (t + 2t^3 + 9t^5)\, dt = [\tfrac{1}{2}t^2 + \tfrac{1}{2}t^4 + \tfrac{3}{2}t^6]_0^1 = \tfrac{5}{2}$$

(*c*) From O to A: $y = z = 0$ and $dy = dz = 0$, and x varies from 0 to 1.
From A to B: $x = 1$, $z = 0$, $dx = dz = 0$, and y varies from 0 to 1.
From B to C: $x = y = 1$ and $dx = dy = 0$, and z varies from 0 to 1.

Now, for the distance from O to A, $W_1 = \int_0^1 x\, dx = \frac{1}{2}$; for the distance from A to B, $W_2 = \int_0^1 y\, dy = \frac{1}{2}$; and for the distance from B to C, $W_3 = \int_0^1 (z + 1)\, dz = \frac{3}{2}$. Thus, $W = W_1 + W_2 + W_3 = \frac{5}{2}$.

In general, the value of a line integral depends upon the path of integration. Here is an example of one which does not, that is, one which is independent of the path. It can be shown that a line integral $\int_C (f_1\, dx + f_2\, dy + f_3\, dz)$ is independent of the path if there exists a function $\phi(x, y, z)$ such that $d\phi = f_1\, dx + f_2\, dy + f_3\, dz$. In this problem the integrand is
$$(x + yz)\, dx + (y + xz)\, dy + (z + xy)\, dz = d[\tfrac{1}{2}(x^2 + y^2 + z^2) + xyz]$$

Supplementary Problems

12. Find ds/dt and $d^2\mathbf{s}/dt^2$, given (*a*) $\mathbf{s} = (t+1)\mathbf{i} + (t^2+t+1)\mathbf{j} + (t^3+t^2+t+1)\mathbf{k}$ and (*b*) $\mathbf{s} = \mathbf{i}e^t \cos 2t + \mathbf{j}e^t \sin 2t + t^2\mathbf{k}$.

   Ans. (*a*) $\mathbf{i} + (2t+1)\mathbf{j} + (3t^2+2t+1)\mathbf{k}$, $2\mathbf{j} + (6t+2)\mathbf{k}$; (*b*) $e^t(\cos 2t - 2\sin 2t)\mathbf{i} + e^t(\sin 2t + 2\cos 2t)\mathbf{j} + 2t\mathbf{k}$, $e^t(-4\sin 2t - 3\cos 2t)\mathbf{i} + e^t(-3\sin 2t + 4\cos 2t)\mathbf{j} + 2\mathbf{k}$

13. Given $\mathbf{a} = u\mathbf{i} + u^2\mathbf{j} + u^3\mathbf{k}$, $\mathbf{b} = \mathbf{i}\cos u + \mathbf{j}\sin u$, and $\mathbf{c} = 3u^2\mathbf{i} - 4u\mathbf{k}$. First compute $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, and find the derivative of each. Then find the derivatives using the formulas.

14. A particle moves along the curve $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$, where t is time. Find (*a*) the magnitudes of its velocity and acceleration at time $t = 1$; (*b*) the components of velocity and acceleration at time $t = 1$ in the direction $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. *Ans.* (*a*) $|\mathbf{v}| = 3\sqrt{5}$, $|\mathbf{a}| = 2\sqrt{19}$; (*b*) 6, $\frac{22}{3}$

15. Using vector methods, find the equations of the tangent line and normal plane to the curves of Problem 11 of Chapter 66.

16. Solve Problem 12 of Chapter 66 using vector methods.

17. Show that the surfaces $x = u$, $y = 5u - 3v^2$, $z = v$ and $x = u$, $y = v$, $z = \dfrac{uv}{4u - v}$ are perpendicular at $P(1, 2, 1)$.

18. Using vector methods, find the equations of the tangent plane and normal line to the surface
   (*a*) $x = u$, $y = v$, $z = uv$ at the point $(u, v) = (3, -4)$
   (*b*) $x = u$, $y = v$, $z = u^2 - v^2$ at the point $(u, v) = (2, 1)$

   Ans. (*a*) $4X - 3Y + Z - 12 = 0$, $\dfrac{X-3}{-4} = \dfrac{Y+4}{3} = \dfrac{Z+12}{-1}$; (*b*) $4X - 2Y - Z - 3 = 0$, $\dfrac{X-2}{-4} = \dfrac{Y-1}{2} = \dfrac{Z-3}{1}$

19. (*a*) Find the equations of the osculating and rectifying planes to the curve of Problem 2 at the given point.
   (*b*) Find the equations of the osculating, normal, and rectifying planes to $x = 2t - t^2$, $y = t^2$, $z = 2t + t^2$ at $t = 1$.

   Ans. (*a*) $3X - 3Y + Z - 1 = 0$, $11X + 8Y - 9Z - 10 = 0$; (*b*) $X + 2Y - Z = 0$, $Y + 2Z - 7 = 0$, $5X - 2Y + Z - 6 = 0$

20. Show that the equation of the osculating plane to a space curve at P is given by

$$(\mathbf{R} - \mathbf{r})\cdot\left(\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right) = 0$$

21. Solve Problems 16 and 17 of Chapter 67, using vector methods.

22. Find $\displaystyle\int_a^b \mathbf{F}(u)\,du$, given

   (*a*) $\mathbf{F}(u) = u^3\mathbf{i} + (3u^2 - 2u)\mathbf{j} + 3\mathbf{k}$; $a = 0$, $b = 2$ (*b*) $\mathbf{F}(u) = e^u\mathbf{i} + e^{-2u}\mathbf{j} + u\mathbf{k}$; $a = 0$, $b = 1$

   Ans. (*a*) $4\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$; (*b*) $(e - 1)\mathbf{i} + \frac{1}{2}(1 - e^{-2})\mathbf{j} + \frac{1}{2}\mathbf{k}$

23. The acceleration of a particle at any time t is given by $\mathbf{a} = d\mathbf{v}/dt = (t+1)\mathbf{i} + t^2\mathbf{j} + (t^2 - 2)\mathbf{k}$. If at $t = 0$, the displacement is $\mathbf{r} = 0$ and the velocity is $\mathbf{v} = \mathbf{i} - \mathbf{k}$, find $\mathbf{v}$ and $\mathbf{r}$ at any time t.

   Ans. $\mathbf{v} = (\frac{1}{2}t^2 + t + 1)\mathbf{i} + \frac{1}{3}t^3\mathbf{j} + (\frac{1}{3}t^3 - 2t - 1)\mathbf{k}$; $\mathbf{r} = (\frac{1}{6}t^3 + \frac{1}{2}t^2 + t)\mathbf{i} + \frac{1}{12}t^4\mathbf{j} + (\frac{1}{12}t^4 - t^2 - t)\mathbf{k}$

24. In each of the following, find the work done by the given force $\mathbf{F}$ in moving a particle from $O(0,0,0)$ to $C(1,1,1)$ along (1) the straight line $x = y = z$, (2) the curve $x = t$, $y = t^2$, $z = t^3$, and (3) the straight lines from O to $A(1,0,0)$, A to $B(1,1,0)$, and B to C.
(*a*) $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + 3x\mathbf{k}$
(*b*) $\mathbf{F} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$
(*c*) $\mathbf{F} = (x + xyz)\mathbf{i} + (y + x^2z)\mathbf{j} + (z + x^2y)\mathbf{k}$

Ans. (*a*) 3; (*b*) 3; (*c*) $\frac{9}{4}$, $\frac{33}{14}$, $\frac{5}{2}$

25. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show that (*a*) $\operatorname{div} \mathbf{r} = 3$ and (*b*) $\operatorname{curl} \mathbf{r} = 0$.

26. If $f = f(x, y, z)$ has partial derivatives of order at least two, show that (*a*) $\nabla \times \nabla f = 0$; (*b*) $\nabla \cdot (\nabla \times f) = 0$; and (*c*) $\nabla \cdot \nabla f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$.

Chapter 69

Double and Iterated Integrals

THE (SIMPLE) INTEGRAL $\int_a^b f(x)\,dx$ of a function $y = f(x)$ that is continuous over the finite interval $a \le x \le b$ of the x axis was defined in Chapter 38. Recall that

1. The interval $a \le x \le b$ was divided into n subintervals $h_1, h_2, \ldots, h_n$ of respective lengths $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$ with λ_n the greatest of the $\Delta_k x$.
2. Points x_1 in h_1, x_2 in $h_2, \ldots, x_n$ in h_n were selected, and the sum $\sum_{k=1}^{n} f(x_k)\,\Delta_k x$ formed.
3. The interval was further subdivided in such a manner that $\lambda_n \to 0$ as $n \to +\infty$.
4. We defined $\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x$.

THE DOUBLE INTEGRAL. Consider a function $z = f(x, y)$ continuous over a finite region R of the xOy plane. Let this region be subdivided (see Fig. 69-1) into n subregions $R_1, R_2, \ldots, R_n$ of respective areas $\Delta_1 A, \Delta_2 A, \ldots, \Delta_n A$. In each subregion R_k, select a point $P_k(x_k, y_k)$ and form the sum

$$\sum_{k=1}^{n} f(x_k, y_k)\,\Delta_k A = f(x_1, y_1)\,\Delta_1 A + f(x_2, y_2)\,\Delta_2 A + \cdots + f(x_n, y_n)\,\Delta_n A \tag{69.1}$$

Now, defining the diameter of a subregion to be the greatest distance between any two points within or on its boundary, and denoting by λ_n the maximum diameter of the subregions, suppose the number of subregions to be increased in such a manner that $\lambda_n \to 0$ as $n \to +\infty$. Then the *double integral* of the function $f(x, y)$ over the region R is defined as

$$\iint_R f(x, y)\,dA = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta_k A \tag{69.2}$$

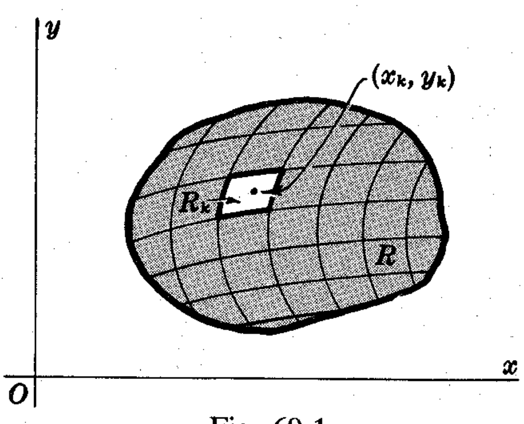

Fig. 69-1

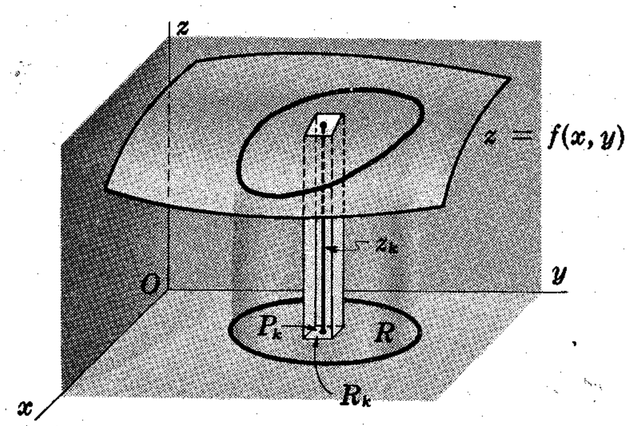

Fig. 69-2

When $z = f(x, y)$ is nonnegative over the region R, as in Fig. 69-2, the double integral (*69.2*) may be interpreted as a volume. Any term $f(x_k, y_k)\,\Delta_k A$ of (*69.1*) gives the volume of a vertical column whose parallel bases are of area $\Delta_k A$ and whose altitude is the distance z_k measured along the vertical from the selected point P_k to the surface $z = f(x, y)$. This, in turn, may be taken as an approximation of the volume of the vertical column whose lower base is the subregion R_k and whose upper base is the projection of R_k on the surface. Thus, (*69.1*) is an approximation of the volume "under the surface" (that is, the volume with lower base in the

xOy plane and upper base in the surface generated by moving a line parallel to the z axis along the boundary of R), and, intuitively, at least, (*69.2*) is the measure of this volume.

The evaluation of even the simplest double integral by direct summation is difficult and will not be attempted here.

THE ITERATED INTEGRAL. Consider a volume defined as above, and assume that the boundary of R is such that no line parallel to the x axis or to the y axis cuts it in more than two points. Draw (see Fig. 69-3) the tangents $x = a$ and $x = b$ to the boundary with points of tangency K and L, and the tangents $y = c$ and $y = d$ with points of tangency M and N. Let the equation of the plane arc LMK be $y = g_1(x)$, and that of the plane arc LNK be $y = g_2(x)$.

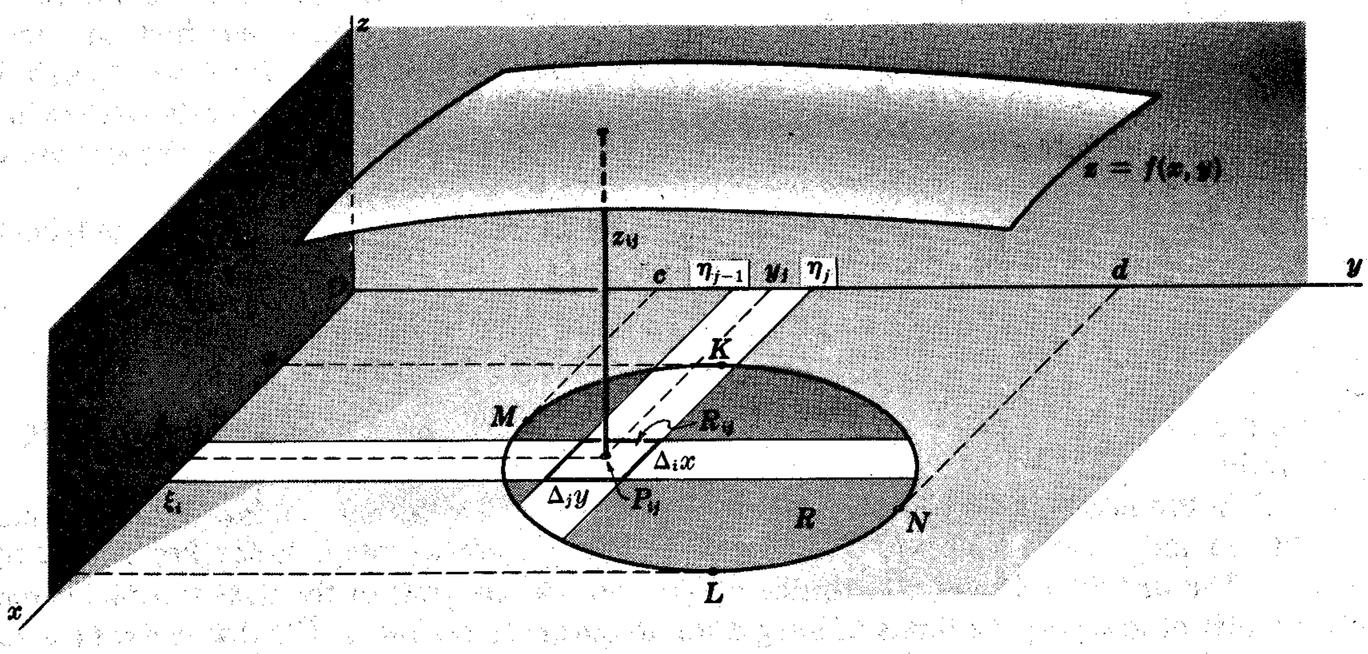

Fig. 69-3

Divide the interval $a \le x \le b$ into m subintervals $h_1, h_2, \ldots, h_m$ of respective lengths $\Delta_1 x$, $\Delta_2 x, \ldots, \Delta_m x$ by the insertion of points $x = \xi_1$, $x = \xi_2, \ldots, x = \xi_{m-1}$ (as in Chapter 38), and divide the interval $c \le y \le d$ into n subintervals $k_1, k_2, \ldots, k_n$ of respective lengths $\Delta_1 y$, $\Delta_2 y, \ldots, \Delta_n y$ by the insertion of points $y = \eta_1$, $y = \eta_2, \ldots, y = \eta_{n-1}$. Denote by λ_m the greatest $\Delta_i x$, and by μ_n the greatest $\Delta_j y$. Draw in the parallel lines $x = \xi_1, x = \xi_2, \ldots, x = \xi_{m-1}$ and the parallel lines $y = \eta_1$, $y = \eta_2, \ldots, y = \eta_{n-1}$, thus dividing the region R into a set of rectangles R_{ij} of areas $\Delta_i x\, \Delta_j y$ plus a set of nonrectangles that we shall ignore. On each subinterval h_i select a point $x = x_i$, and on each subinterval k_j select a point $y = y_j$, thereby determining in each subregion R_{ij} a point $P_{ij}(x_i, y_j)$. With each subregion R_{ij}, associate by means of the equation of the surface a number $z_{ij} = f(x_i, y_j)$, and form the sum

$$\sum_{\substack{i=1,2,\ldots,m\\ j=1,2,\ldots,n}} f(x_i, y_j)\,\Delta_i x\,\Delta_j y \tag{69.3}$$

Now (*69.3*) is merely a special case of (*69.1*), so if the number of rectangles is indefinitely increased in such a manner that both $\lambda_m \to 0$ and $\mu_n \to 0$, the limit of (*69.3*) should be equal to the double integral (*69.2*).

In effecting this limit, let us first choose one of the subintervals, say h_i, and form the sum

$$\left[\sum_{j=1}^{n} f(x_i, y_j)\,\Delta_j y\right]\Delta_i x \quad (i \text{ fixed})$$

of the contributions of all rectangles having h_i as one dimension, that is, the contributions of all rectangles lying in the ith column. When $n \to +\infty$, $\mu_n \to 0$ and

$$\lim_{n\to+\infty}\left[\sum_{j=1}^{n} f(x_i, y_j)\,\Delta_j y\right]\Delta_i x = \left[\int_{g_1(x_i)}^{g_2(x_1)} f(x_i, y)\,dy\right]\Delta_i x = \phi(x_i)\,\Delta_i x$$

Now summing over the m columns and letting $m \to +\infty$, we have

$$\lim_{m\to+\infty}\sum_{i=1}^{m}\phi(x_i)\,\Delta_i x = \int_a^b \phi(x)\,dx = \int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\right]dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx \qquad (69.4)$$

Although we shall not use the brackets hereafter, it must be clearly understood that *(69.4)* calls for the evaluation of two simple definite integrals in a prescribed order: first, the integral of $f(x, y)$ with respect to y (considering x as a constant) from $y = g_1(x)$, the lower boundary of R, to $y = g_2(x)$, the upper boundary of R, and then the integral of this result with respect to x from the abscissa $x = a$ of the leftmost point of R to the abscissa $x = b$ of the rightmost point of R. The integral *(69.4)* is called an *iterated* or *repeated integral.*

It will be left as an exercise to sum first for the contributions of the rectangles lying in each row and then over all the rows to obtain the equivalent iterated integral

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy \qquad (69.5)$$

where $x = h_1(y)$ and $x = h_2(y)$ are the equations of the plane arcs MKN and MLN, respectively.

In Problem 1 it is shown by a different procedure that the iterated integral *(69.4)* measures the volume under discussion. For the evaluation of iterated integrals see Problems 2 to 6.

The principal difficulty in setting up the iterated integrals of the next several chapters will be that of inserting the limits of integration to cover the region R. The discussion here assumed the simplest of regions; more complex regions are considered in Problems 7 to 9.

Solved Problems

1. Let $z = f(x, y)$ be nonnegative and continuous over the region R of the plane xOy whose boundary consists of the arcs of two curves $y = g_1(x)$ and $y = g_2(x)$ intersecting in the points K and L, as in Fig. 69-4. Find a formula for the volume V under the surface $z = f(x, y)$.

Let the section of this volume cut by a plane $x = x_i$, where $a < x_i < b$, meet the boundary of R in the points $S(x_i, g_1(x_i))$ and $T(x_i, g_2(x_i))$, and the surface $z = f(x, y)$ in the arc UV along which $z = f(x_i, y)$. The area of this section $STUV$ is given by

$$A(x_i) = \int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy$$

Thus, the areas of cross sections of the volume cut by planes parallel to the yOz plane are known functions $A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy$ of x, where x is the distance of the sectioning plane from the origin. By Chapter 42, the required volume is given by

$$V = \int_a^b A(x)\,dx = \int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\right]dx$$

This is the iterated integral of *(69.4)*.

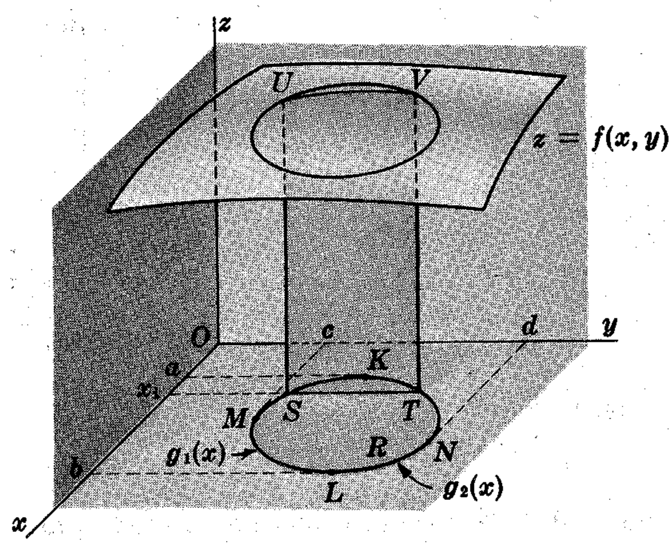

Fig. 69-4

In Problems 2 to 6, evaluate the integral at the left.

2. $\displaystyle\int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 [y]_{x^2}^{x}\,dx = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$

3. $\displaystyle\int_1^2 \int_y^{3y} (x + y)\,dx\,dy = \int_1^2 [\tfrac{1}{2}x^2 + xy]_y^{3y}\,dy = \int_1^2 6y^2\,dy = [2y^3]_1^2 = 14$

4. $\displaystyle\int_{-1}^{2} \int_{2x^2-2}^{x^2+x} x\,dy\,dx = \int_{-1}^{2} [xy]_{2x^2-2}^{x^2+x}\,dx = \int_{-1}^{2} (x^3 + x^2 - 2x^3 + 2x)\,dx = \tfrac{9}{4}$

5. $\displaystyle\int_0^{\pi} \int_0^{\cos\theta} \rho\sin\theta\,d\rho\,d\theta = \int_0^{\pi} [\tfrac{1}{2}\rho^2 \sin\theta]_0^{\cos\theta}\,d\theta = \tfrac{1}{2}\int_0^{\pi} \cos^2\theta\sin\theta\,d\theta = [-\tfrac{1}{6}\cos^3\theta]_0^{\pi} = \tfrac{1}{3}$

6. $\displaystyle\int_0^{\pi/2} \int_2^{4\cos\theta} \rho^3\,d\rho\,d\theta = \int_0^{\pi/2} \left[\frac{1}{4}\rho^4\right]_2^{4\cos\theta} d\theta = \int_0^{\pi/2} (64\cos^4\theta - 4)\,d\theta$

$$= \left[64\left(\frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right) - 4\theta\right]_0^{\pi/2} = 10\pi$$

7. Evaluate $\displaystyle\iint_R dA$, where R is the region in the first quadrant bounded by the semicubical parabola $y^2 = x^3$ and the line $y = x$.

The line and parabola intersect in the points $(0, 0)$ and $(1, 1)$ which establish the extreme values of x and y on the region R.

Solution 1 (Fig. 69-5): Integrating first over a horizontal strip, that is, with respect to x from $x = y$ (the line) to $x = y^{2/3}$ (the parabola), and then with respect to y from $y = 0$ to $y = 1$, we get

$$\iint_R dA = \int_0^1 \int_y^{y^{2/3}} dx\,dy = \int_0^1 (y^{2/3} - y)\,dy = [\tfrac{3}{5}y^{5/3} - \tfrac{1}{2}y^2]_0^1 = \tfrac{1}{10}$$

Solution 2 (Fig. 69-6): Integrating first over a vertical strip, that is, with respect to y from $y = x^{3/2}$ (the parabola) to $y = x$ (the line), and then with respect to x from $x = 0$ to $x = 1$, we obtain

$$\iint_R dA = \int_0^1 \int_{x^{3/2}}^{x} dy\,dx = \int_0^1 (x - x^{3/2})\,dx = [\tfrac{1}{2}x^2 - \tfrac{2}{5}x^{5/2}]_0^1 = \tfrac{1}{10}$$

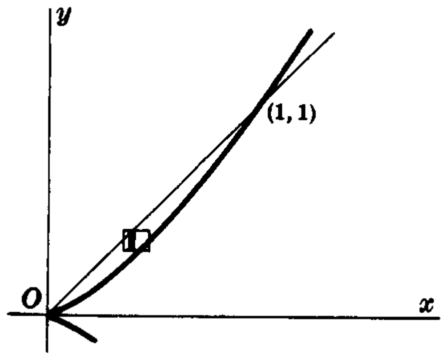

Fig. 69-5

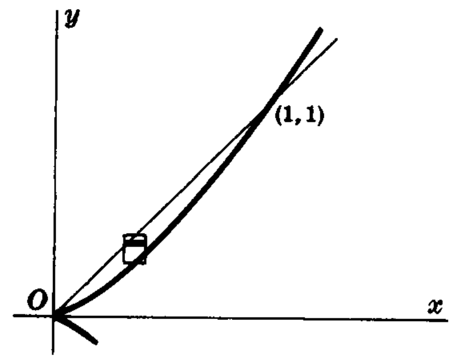

Fig. 69-6

8. Evaluate $\iint_R dA$ where R is the region between $y = 2x$ and $y = x^2$ lying to the left of $x = 1$.

Integrating first over the vertical strip (see Fig. 69-7), we have

$$\iint_R dA = \int_0^1 \int_{x^2}^{2x} dy\,dx = \int_0^1 (2x - x^2)\,dx = \tfrac{2}{3}$$

When horizontal strips are used (see Fig. 69-8), two iterated integrals are necessary. Let R_1 denote the part of R lying below AB, and R_2 the part above AB. Then

$$\iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{y/2}^{\sqrt{y}} dx\,dy + \int_1^2 \int_{y/2}^{1} dx\,dy = \tfrac{5}{12} + \tfrac{1}{4} = \tfrac{2}{3}$$

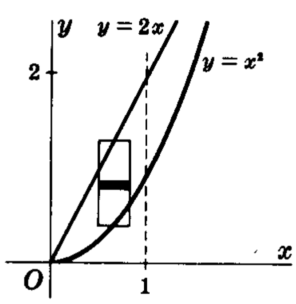

Fig. 69-7

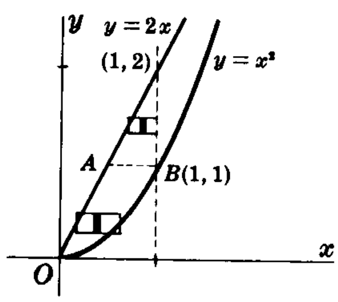

Fig. 69-8

9. Evaluate $\iint_R x^2\,dA$ where R is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$, and $x = 8$. (See Fig. 69-9.)

It is evident from Fig. 69-9 that R must be separated into two regions, and an iterated integral evaluated for each. Let R_1 denote the part of R lying above the line $y = 2$, and R_2 the part below that line. Then

$$\iint_R x^2\,dA = \iint_{R_1} x^2\,dA + \iint_{R_2} x^2\,dA = \int_2^4 \int_y^{16/y} x^2\,dx\,dy + \int_0^2 \int_y^8 x^2\,dx\,dy$$

$$= \frac{1}{3}\int_2^4 \left(\frac{16^3}{y^3} - y^3\right) dy + \frac{1}{3}\int_0^2 (8^3 - y^3)\,dy = 448$$

As an exercise, you might separate R with the line $x = 4$ and obtain

$$\iint_R x^2\,dA = \int_0^4 \int_0^x x^2\,dy\,dx + \int_4^8 \int_0^{16/x} x^2\,dy\,dx$$

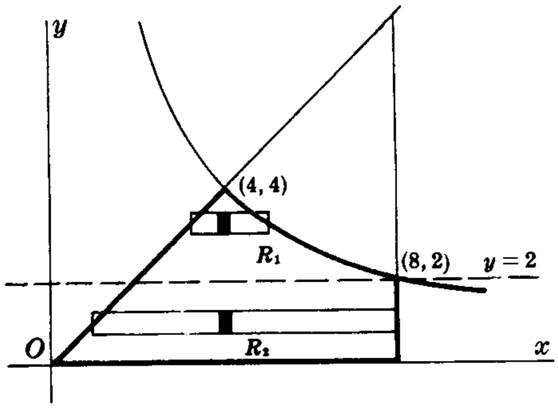

Fig. 69-9

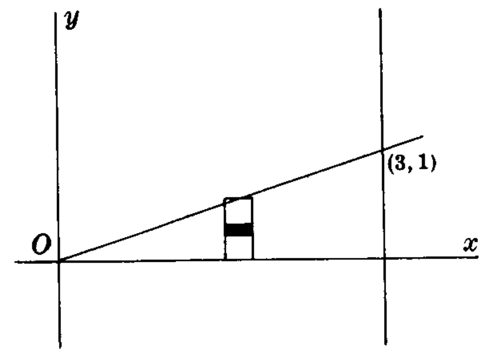

Fig. 69-10

10. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$ by first reversing the order of integration.

The given integral cannot be evaluated directly, since $\int e^{x^2}\,dx$ is not an elementary function. The region R of integration (see Fig. 69-10) is bounded by the lines $x = 3y$, $x = 3$, and $y = 0$. To reverse the order of integration, first integrate with respect to y from $y = 0$ to $y = x/3$, and then with respect to x from $x = 0$ to $x = 3$. Thus,

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 [e^{x^2}y]_0^{x/3}\,dx$$

$$= \tfrac{1}{3}\int_0^3 e^{x^2}x\,dx = [\tfrac{1}{6}e^{x^2}]_0^3 = \tfrac{1}{6}(e^9 - 1)$$

Supplementary Problems

11. Evaluate the iterated integral at the left:

(a) $\int_0^1 \int_1^2 dx\,dy = 1$

(b) $\int_1^2 \int_0^3 (x + y)\,dx\,dy = 9$

(c) $\int_2^4 \int_1^2 (x^2 + y^2)\,dy\,dx = \frac{70}{3}$

(d) $\int_0^1 \int_{x^2}^x xy^2\,dy\,dx = \frac{1}{40}$

(e) $\int_1^2 \int_0^{y^{3/2}} x/y^2\,dx\,dy = \frac{3}{4}$

(f) $\int_0^1 \int_x^{\sqrt{x}} (y + y^3)\,dy\,dx = \frac{7}{60}$

(g) $\int_0^1 \int_0^{x^2} xe^y\,dy\,dx = \frac{1}{2}e - 1$

(h) $\int_2^4 \int_y^{8-y} y\,dx\,dy = \frac{32}{3}$

(i) $\int_0^{\text{Arctan } 3/2} \int_0^{2\sec\theta} \rho\,d\rho\,d\theta = 3$

(j) $\int_0^{\pi/2} \int_0^2 \rho^2\cos\theta\,d\rho\,d\theta = \frac{8}{3}$

(k) $\int_0^{\pi/4} \int_0^{\tan\theta\sec\theta} \rho^3\cos^2\theta\,d\rho\,d\theta = \frac{1}{20}$

(l) $\int_0^{2\pi} \int_0^{1-\cos\theta} \rho^3\cos^2\theta\,d\rho\,d\theta = \frac{49}{32}\pi$

12. Using an iterated integral, evaluate each of the following double integrals. When feasible, evaluate the iterated integral in both orders.

(*a*) x over the region bounded by $y = x^2$ and $y = x^3$ *Ans.* $\frac{1}{20}$

(*b*) y over the region of part (*a*) *Ans.* $\frac{1}{35}$

(*c*) x^2 over the region bounded by $y = x$, $y = 2x$, and $x = 2$ *Ans.* 4

(*d*) 1 over each first-quadrant region bounded by $2y = x^2$, $y = 3x$, and $x + y = 4$ *Ans.* $\frac{8}{3}$; $\frac{46}{3}$

(*e*) y over the region above $y = 0$ bounded by $y^2 = 4x$ and $y^2 = 5 - x$ *Ans.* 5

(*f*) $\dfrac{1}{\sqrt{2y - y^2}}$ over the region in the first quadrant bounded by $x^2 = 4 - 2y$ *Ans.* 4

13. In Problem 11(*a*) to (*h*), reverse the order of integration and evaluate the resulting iterated integral.

Chapter 70

Centroids and Moments of Inertia of Plane Areas

PLANE AREA BY DOUBLE INTEGRATION. If $f(x, y) = 1$, the double integral of Chapter 69 becomes $\iint_R dA$. In cubic units, this measures the volume of a cylinder of unit height; in square units, it measures the area of the region R. (See Problems 1 and 2.)

In polar coordinates, $A = \iint_R dA = \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} \rho \, d\rho \, d\theta$, where $\theta = \alpha$, $\theta = \beta$, $\rho_1(\theta)$, and $\rho_2(\theta)$ are chosen to cover the region R. (See Problems 3 to 5.)

CENTROIDS. The coordinates $(\bar{x}, \bar{y})$ of the centroid of a plane region R of area $A = \iint_R dA$ satisfy the relations

$$A\bar{x} = M_y \qquad \text{and} \qquad A\bar{y} = M_x$$

or

$$\bar{x} \iint_R dA = \iint_R x \, dA \qquad \text{and} \qquad \bar{y} \iint_R dA = \iint_R y \, dA$$

(See Problems 6 to 9.)

THE MOMENTS OF INERTIA of a plane region R with respect to the coordinate axes are given by

$$I_x = \iint_R y^2 \, dA \qquad \text{and} \qquad I_y = \iint_R x^2 \, dA$$

The polar moment of inertia (the moment of inertia with respect to a line through the origin and perpendicular to the plane of the area) of a plane region R is given by

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2) \, dA$$

(See Problems 10 to 12.)

Solved Problems

1. Find the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.

Using vertical strips (see Fig. 70-1), we have

$$A = \int_{-1}^{3} \int_{x^2}^{2x+3} dy \, dx = \int_{-1}^{3} (2x + 3 - x^2) \, dx = 32/3 \text{ square units}$$

2. Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$.

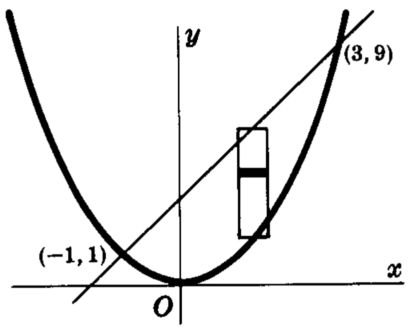

Fig. 70-1

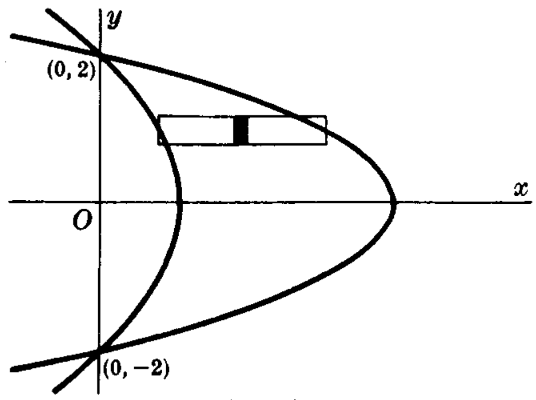

Fig. 70-2

Using horizontal strips (Fig. 70-2) and taking advantage of symmetry, we have

$$A = 2\int_0^2 \int_{1-y^2/4}^{4-y^2} dx\,dy = 2\int_0^2 [(4-y^2)-(1-\tfrac{1}{4}y^2)]\,dy$$
$$= 6\int_0^2 (1-\tfrac{1}{4}y^2)\,dy = 8 \text{ square units}$$

3. Find the area outside the circle $\rho = 2$ and inside the cardioid $\rho = 2(1+\cos\theta)$.

Owing to symmetry (see Fig. 70-3), the required area is twice that swept over as θ varies from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. Thus,

$$A = 2\int_0^{\pi/2}\int_2^{2(1+\cos\theta)} \rho\,d\rho\,d\theta = 2\int_0^{\pi/2} [\tfrac{1}{2}\rho^2]_2^{2(1+\cos\theta)}\,d\theta = 4\int_0^{\pi/2} (2\cos\theta + \cos^2\theta)\,d\theta$$
$$= 4[2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta]_0^{\pi/2} = (\pi + 8) \text{ square units}$$

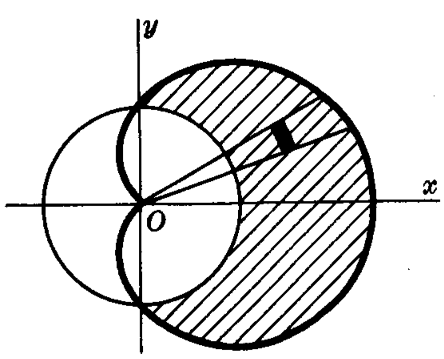

Fig. 70-3

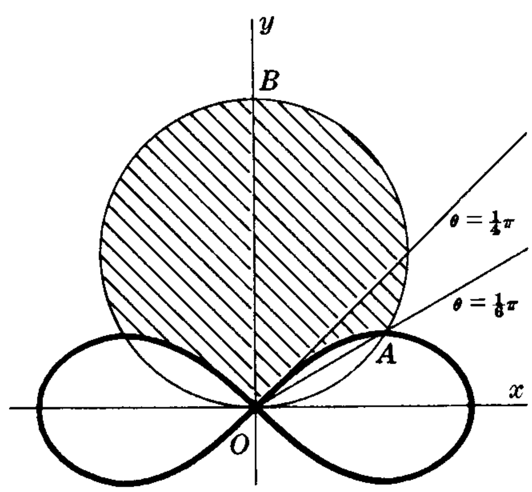

Fig. 70-4

4. Find the area inside the circle $\rho = 4\sin\theta$ and outside the lemniscate $\rho^2 = 8\cos 2\theta$.

The required area is twice that in the first quadrant bounded by the two curves and the line $\theta = \frac{1}{2}\pi$. Note in Fig. 70-4 that the arc AO of the lemniscate is described as θ varies from $\theta = \pi/6$ to $\theta = \pi/4$, while the arc AB of the circle is described as θ varies from $\theta = \pi/6$ to $\theta = \pi/2$. This area must then be considered as two regions, one below and one above the line $\theta = \pi/4$. Thus,

$$A = 2\int_{\pi/6}^{\pi/4}\int_{2\sqrt{2\cos 2\theta}}^{4\sin\theta} \rho\,d\rho\,d\theta + 2\int_{\pi/4}^{\pi/2}\int_0^{4\sin\theta} \rho\,d\rho\,d\theta$$
$$= \int_{\pi/6}^{\pi/4} (16\sin^2\theta - 8\cos 2\theta)\,d\theta + \int_{\pi/4}^{\pi/2} 16\sin^2\theta\,d\theta$$
$$= (\tfrac{8}{3}\pi + 4\sqrt{3} - 4) \text{ square units}$$

5. Evaluate $N = \int_0^{+\infty} e^{-x^2}\,dx$. (See Fig. 70-5.)

Since $\int_0^{+\infty} e^{-x^2}\,dx = \int_0^{+\infty} e^{-y^2}\,dy$, we have

$$N^2 = \int_0^{+\infty} e^{-x^2}\,dx \int_0^{+\infty} e^{-y^2}\,dy = \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy = \iint_R e^{-(x^2+y^2)}\,dA$$

Changing to polar coordinates ($x^2 + y^2 = \rho^2$, $dA = \rho\,d\rho\,d\theta$) yields

$$N^2 = \int_0^{\pi/2}\int_0^{+\infty} e^{-\rho^2}\rho\,d\rho\,d\theta = \int_0^{\pi/2} \lim_{a\to+\infty}\left[-\frac{1}{2}e^{-\rho^2}\right]_0^a d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

and $N = \sqrt{\pi}/2$.

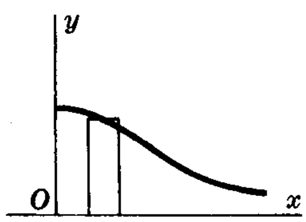

Fig. 70-5

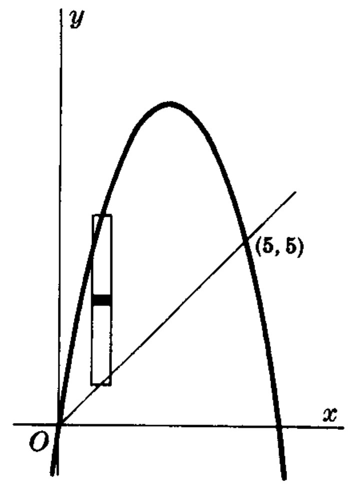

Fig. 70-6

6. Find the centroid of the plane area bounded by the parabola $y = 6x - x^2$ and the line $y = x$. (See Fig. 70-6.)

$$A = \iint_R dA = \int_0^5\int_x^{6x-x^2} dy\,dx = \int_0^5 (5x - x^2)\,dx = \tfrac{125}{6}$$

$$M_y = \iint_R x\,dA = \int_0^5\int_x^{6x-x^2} x\,dy\,dx = \int_0^5 (5x^2 - x^3)\,dx = \tfrac{625}{12}$$

$$M_x = \iint_R y\,dA = \int_0^5\int_x^{6x-x^2} y\,dy\,dx = \tfrac{1}{2}\int_0^5 [(6x - x^2)^2 - x^2]\,dx = \tfrac{625}{6}$$

Hence, $\bar{x} = M_y/A = \frac{5}{2}$, $\bar{y} = M_x/A = 5$, and the coordinates of the centroid are $(\frac{5}{2}, 5)$.

7. Find the centroid of the plane area bounded by the parabolas $y = 2x - x^2$ and $y = 3x^2 - 6x$. (See Fig. 70-7.)

$$A = \iint_R dA = \int_0^2\int_{3x^2-6x}^{2x-x^2} dy\,dx = \int_0^2 (8x - 4x^2)\,dx = \tfrac{16}{3}$$

$$M_y = \iint_R x\,dA = \int_0^2\int_{3x^2-6x}^{2x-x^2} x\,dy\,dx = \int_0^2 (8x^2 - 4x^3)\,dx = \tfrac{16}{3}$$

$$M_x = \iint_R y\,dA = \int_0^2\int_{3x^2-6x}^{2x-x^2} y\,dy\,dx = \tfrac{1}{2}\int_0^2 [(2x - x^2)^2 - (3x^2 - 6x)^2]\,dx = -\tfrac{64}{15}$$

Hence, $\bar{x} = M_y/A = 1$, $\bar{y} = M_x/A = -\frac{4}{5}$, and the centroid is $(1, -\frac{4}{5})$.

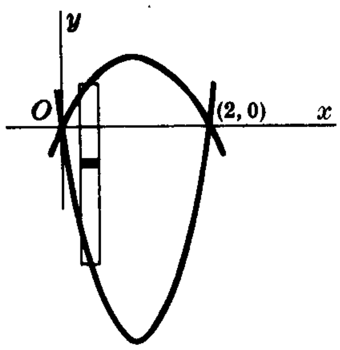

Fig. 70-7

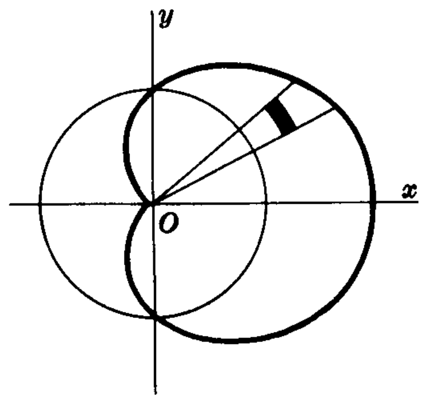

Fig. 70-8

8. Find the centroid of the plane area outside the circle $\rho = 1$ and inside the cardioid $\rho = 1 + \cos\theta$.

From Fig. 70-8 it is evident that $\bar{y} = 0$ and that $\bar{x}$ is the same whether computed for the given area or for the half lying above the polar axis. For the latter area,

$$A = \iint_R dA = \int_0^{\pi/2}\int_1^{1+\cos\theta} \rho\, d\rho\, d\theta = \frac{1}{2}\int_0^{\pi/2} [(1+\cos\theta)^2 - 1^2]\, d\theta = \frac{\pi+8}{8}$$

$$M_y = \iint_R x\, dA = \int_0^{\pi/2}\int_1^{1+\cos\theta} (\rho\cos\theta)\rho\, d\rho\, d\theta = \frac{1}{3}\int_0^{\pi/2} (3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\, d\theta$$

$$= \frac{1}{3}\left[\frac{3}{2}\theta + \frac{3}{4}\sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right]_0^{\pi/2} = \frac{15\pi+32}{48}$$

The coordinates of the centroid are $\left(\dfrac{15\pi+32}{6(\pi+8)}, 0\right)$.

9. Find the centroid of the area inside $\rho = \sin\theta$ and outside $\rho = 1 - \cos\theta$. (See Fig. 70-9.)

$$A = \iint_R dA = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} \rho\, d\rho\, d\theta = \frac{1}{2}\int_0^{\pi/2} (2\cos\theta - 1 - \cos 2\theta)\, d\theta = \frac{4-\pi}{4}$$

$$M_y = \iint_R x\, dA = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} (\rho\cos\theta)\rho\, d\rho\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (\sin^3\theta - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta)\cos\theta\, d\theta = \frac{15\pi-44}{48}$$

$$M_x = \iint_R y\, dA = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} (\rho\sin\theta)\rho\, d\rho\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (\sin^3\theta - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta)\sin\theta\, d\theta = \frac{3\pi-4}{48}$$

The coordinates of the centroid are $\left(\dfrac{15\pi-44}{12(4-\pi)}, \dfrac{3\pi-4}{12(4-\pi)}\right)$.

10. Find I_x, I_y, and I_0 for the area enclosed by the loop of $y^2 = x^2(2-x)$. (See Fig. 70-10.)

$$A = \iint_R dA = 2\int_0^2\int_0^{x\sqrt{2-x}} dy\, dx = 2\int_0^2 x\sqrt{2-x}\, dx$$

$$= -4\int_{\sqrt{2}}^0 (2z^2 - z^4)\, dz = -4\left[\frac{2}{3}z^3 - \frac{1}{5}z^5\right]_{\sqrt{2}}^0 = \frac{32\sqrt{2}}{15}$$

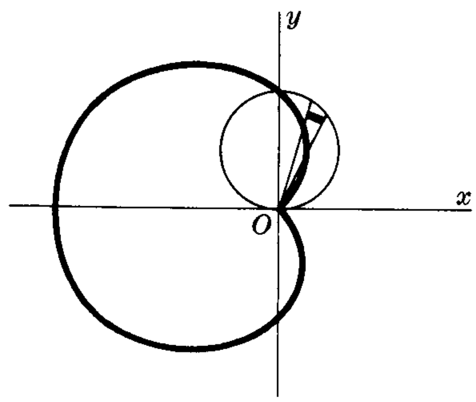

Fig. 70-9

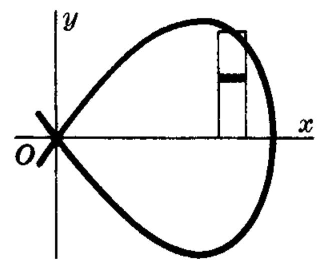

Fig. 70-10

where we have used the transformation $2-x=z^2$. Then

$$I_x = \iint_R y^2\,dA = 2\int_0^2 \int_0^{x\sqrt{2-x}} y^2\,dy\,dx = \frac{2}{3}\int_0^2 x^3(2-x)^{3/2}\,dx$$

$$= -\frac{4}{3}\int_{\sqrt{2}}^0 (2-z^2)^3 z^4\,dz = -\frac{4}{3}\left[\frac{8}{5}z^5 - \frac{12}{7}z^7 + \frac{2}{3}z^9 - \frac{1}{11}z^{11}\right]_{\sqrt{2}}^0 = \frac{2048\sqrt{2}}{3465} = \frac{64}{231}A$$

$$I_y = \iint_R x^2\,dA = 2\int_0^2 \int_0^{x\sqrt{2-x}} x^2\,dy\,dx = 2\int_0^2 x^3\sqrt{2-x}\,dx$$

$$= -4\int_{\sqrt{2}}^0 (2-z^2)^3 z^2\,dz = -4\left[\frac{8}{3}z^3 - \frac{12}{5}z^5 + \frac{6}{7}z^7 - \frac{1}{9}z^9\right]_{\sqrt{2}}^0 = \frac{1024\sqrt{2}}{315} = \frac{32}{21}A$$

$$I_0 = I_x + I_y = \frac{13\,312\sqrt{2}}{3465} = \frac{416}{231}A$$

11. Find I_x, I_y, and I_0 for the first-quadrant area outside the circle $\rho = 2a$ and inside the circle $\rho = 4a\cos\theta$. (See Fig. 70-11.)

$$A = \iint_R dA = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} \rho\,d\rho\,d\theta = \frac{1}{2}\int_0^{\pi/3} [(4a\cos\theta)^2 - (2a)^2]\,d\theta = \frac{2\pi + 3\sqrt{3}}{3}a^2$$

$$I_x = \iint_R y^2\,dA = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} (\rho\sin\theta)^2\rho\,d\rho\,d\theta = \frac{1}{4}\int_0^{\pi/3} \{(4a\cos\theta)^4 - (2a)^4\}\sin^2\theta\,d\theta$$

$$= 4a^4\int_0^{\pi/3} (16\cos^4\theta - 1)\sin^2\theta\,d\theta = \frac{4\pi + 9\sqrt{3}}{6}a^4 = \frac{4\pi + 9\sqrt{3}}{2(2\pi + 3\sqrt{3})}a^2 A$$

$$I_y = \iint_R x^2\,dA = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} (\rho\cos\theta)^2\rho\,d\rho\,d\theta = \frac{12\pi + 11\sqrt{3}}{2}a^4 = \frac{3(12\pi + 11\sqrt{3})}{2(2\pi + 3\sqrt{3})}a^2 A$$

$$I_0 = I_x + I_y = \frac{20\pi + 21\sqrt{3}}{3}a^4 = \frac{20\pi + 21\sqrt{3}}{2\pi + 3\sqrt{3}}a^2 A$$

12. Find I_x, I_y, and I_0 for the area of the circle $\rho = 2(\sin\theta + \cos\theta)$. (See Fig. 70-12.)

Since $x^2 + y^2 = \rho^2$,

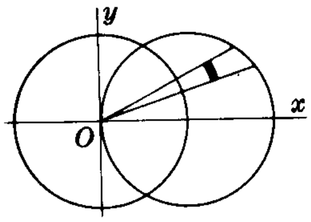

Fig. 70-11

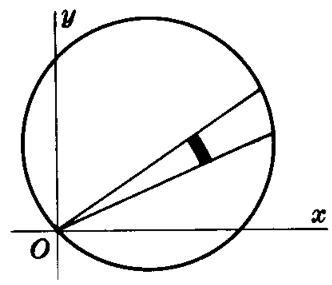

Fig. 70-12

$$I_0 = \iint_R (x^2 + y^2)\, dA = \int_{-\pi/4}^{3\pi/4} \int_0^{2(\sin\theta + \cos\theta)} \rho^2 \rho \, d\rho \, d\theta = 4 \int_{-\pi/4}^{3\pi/4} (\sin\theta + \cos\theta)^4 \, d\theta$$

$$= 4[\tfrac{3}{2}\theta - \cos 2\theta - \tfrac{1}{8}\sin 4\theta]_{-1/4\pi}^{3/4\pi} = 6\pi = 3A$$

It is evident from Fig. 70-12 that $I_x = I_y$. Hence, $I_x = I_y = \frac{1}{2} I_0 = \frac{3}{2} A$.

Supplementary Problems

13. Use double integration to find the area:

(*a*) Bounded by $3x + 4y = 24$, $x = 0$, $y = 0$ *Ans.* 24 square units

(*b*) Bounded by $x + y = 2$, $2y = x + 4$, $y = 0$ *Ans.* 6 square units

(*c*) Bounded by $x^2 = 4y$, $8y = x^2 + 16$ *Ans.* $\frac{32}{3}$ square units

(*d*) Within $\rho = 2(1 - \cos\theta)$ *Ans.* 6π square units

(*e*) Bounded by $\rho = \tan\theta \sec\theta$ and $\theta = \pi/3$ *Ans.* $\frac{1}{2}\sqrt{3}$ square units

(*f*) Outside $\rho = 4$ and inside $\rho = 8\cos\theta$ *Ans.* $8(\frac{2}{3}\pi + \sqrt{3})$ square units

14. Locate the centroid of each of the following areas.

(*a*) The area of Problem 13(*a*) *Ans.* $(\frac{8}{3}, 2)$

(*b*) The first-quadrant area of Problem 13(*c*) *Ans.* $(\frac{3}{2}, \frac{8}{5})$

(*c*) The first-quadrant area bounded by $y^2 = 6x$, $y = 0$, $x = 6$ *Ans.* $(\frac{18}{5}, \frac{9}{4})$

(*d*) The area bounded by $y^2 = 4x$, $x^2 = 5 - 2y$, $x = 0$ *Ans.* $(\frac{13}{40}, \frac{26}{15})$

(*e*) The first-quadrant area bounded by $x^2 - 8y + 4 = 0$, $x^2 = 4y$, $x = 0$ *Ans.* $(\frac{3}{4}, \frac{2}{5})$

(*f*) The area of Problem 13(*e*) *Ans.* $(\frac{1}{2}\sqrt{3}, \frac{6}{5})$

(*g*) The first-quadrant area of Problem 13(*f*) *Ans.* $\left(\dfrac{16\pi + 6\sqrt{3}}{2\pi + 3\sqrt{3}}, \dfrac{22}{2\pi + 3\sqrt{3}}\right)$

15. Verify that $\frac{1}{2}\int_\alpha^\beta [g_2^2(\theta) - g_1^2(\theta)]\, d\theta = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \rho \, d\rho \, d\theta = \iint_R dA$; then infer that

$$\iint_R f(x, y)\, dA = \iint_R f(\rho\cos\theta, \rho\sin\theta)\rho \, d\rho \, d\theta$$

16. Find I_x and I_y for each of the following areas.

(*a*) The area of Problem 13(*a*) *Ans.* $I_x = 6A$; $I_y = \frac{32}{3}A$

(*b*) The area cut from $y^2 = 8x$ by its latus rectum *Ans.* $I_x = \frac{16}{5}A$; $I_y = \frac{12}{7}A$

(*c*) The area bounded by $y = x^2$ and $y = x$ *Ans.* $I_x = \frac{3}{14}A$; $I_y = \frac{3}{10}A$

(*d*) The area bounded by $y = 4x - x^2$ and $y = x$ *Ans.* $I_x = \frac{459}{70}A$; $I_y = \frac{27}{10}A$

17. Find I_x and I_y for one loop of $\rho^2 = \cos 2\theta$. *Ans.* $I_x = \left(\dfrac{\pi}{16} - \dfrac{1}{6}\right)A$; $I_y = \left(\dfrac{\pi}{16} + \dfrac{1}{6}\right)A$

18. Find I_0 for (*a*) the loop of $\rho = \sin 2\theta$ and (*b*) the area enclosed by $\rho = 1 + \cos\theta$. *Ans.* (*a*) $\frac{3}{8}A$; (*b*) $\frac{35}{24}A$

Chapter 71

Volume Under a Surface by Double Integration

THE VOLUME UNDER A SURFACE $z = f(x, y)$ or $z = f(\rho, \theta)$, that is, the volume of a vertical column whose upper base is in the surface and whose lower base is in the xOy plane, is defined by the double integral $V = \iint_R z\, dA$, the region R being the lower base of the column.

Solved Problems

1. Find the volume in the first octant between the planes $z = 0$ and $z = x + y + 2$, and inside the cylinder $x^2 + y^2 = 16$.

From Fig. 71-1, it is evident that $z = x + y + 2$ is to be integrated over a quadrant of the circle $x^2 + y^2 = 16$ in the xOy plane. Hence,

$$V = \iint_R z\, dA = \int_0^4 \int_0^{\sqrt{16-x^2}} (x + y + 2)\, dy\, dx = \int_0^4 \left(x\sqrt{16 - x^2} + 8 - \frac{1}{2}x^2 + 2\sqrt{16 - x^2} \right) dx$$

$$= \left[-\frac{1}{3}(16 - x^2)^{3/2} + 8x - \frac{x^3}{6} + x\sqrt{16 - x^2} + 16 \arcsin \frac{1}{4}x \right]_0^4 = \left(\frac{128}{3} + 8\pi \right) \text{ cubic units}$$

2. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

From Fig. 71-2, it is evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xOy plane. Hence,

$$V = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4 - y)\, dx\, dy = 2\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} (4 - y)\, dx\, dy = 16\pi \text{ cubic units}$$

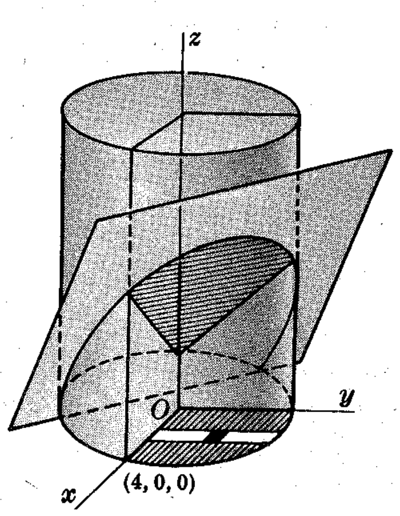

Fig. 71-1

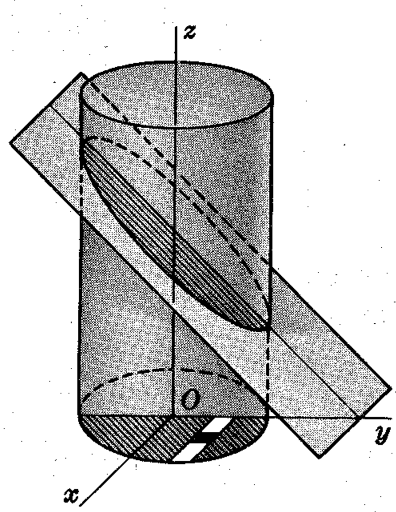

Fig. 71-2

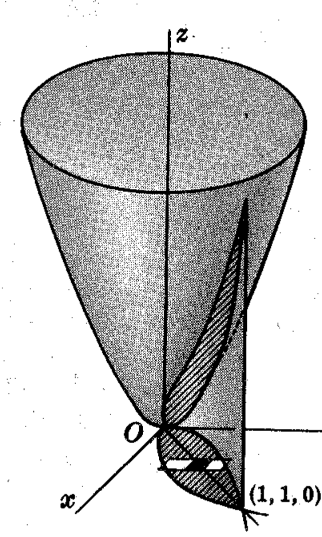

Fig. 71-3

3. Find the volume bounded above by the paraboloid $x^2 + 4y^2 = z$, below by the plane $z = 0$, and laterally by the cylinders $y^2 = x$ and $x^2 = y$. (See Fig. 71-3.)

The required volume is obtained by integrating $z = x^2 + 4y^2$ over the region R common to the parabolas $y^2 = x$ and $x^2 = y$ in the xOy plane. Hence,

$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + 4y^2)\, dy\, dx = \int_0^1 [x^2y + \tfrac{4}{3}y^3]_{x^2}^{\sqrt{x}}\, dx = \tfrac{3}{7} \text{ cubic units}$$

4. Find the volume of one of the wedges cut from the cylinder $4x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = my$. (See Fig. 71-4.)

The volume is obtained by integrating $z = my$ over half the ellipse $4x^2 + y^2 = a^2$. Hence,

$$V = 2\int_0^{a/2} \int_0^{\sqrt{a^2-4x^2}} my\, dy\, dx = m \int_0^{a/2} [y^2]_0^{\sqrt{a^2-4x^2}}\, dx = \frac{ma^3}{3} \text{ cubic units}$$

5. Find the volume bounded by the paraboloid $x^2 + y^2 = 4z$, the cylinder $x^2 + y^2 = 8y$, and the plane $z = 0$. (See Fig. 71-5.)

The required volume is obtained by integrating $z = \frac{1}{4}(x^2 + y^2)$ over the circle $x^2 + y^2 = 8y$. Using cylindrical coordinates, the volume is obtained by integrating $z = \frac{1}{4}\rho^2$ over the circle $\rho = 8 \sin \theta$. Then,

$$V = \iint_R z\, dA = \int_0^{\pi} \int_0^{8 \sin \theta} z\rho\, d\rho\, d\theta = \tfrac{1}{4} \int_0^{\pi} \int_0^{8 \sin \theta} \rho^3\, d\rho\, d\theta$$

$$= \tfrac{1}{16} \int_0^{\pi} [\rho^4]_0^{8 \sin \theta}\, d\theta = 256 \int_0^{\pi} \sin^4 \theta\, d\theta = 96\pi \text{ cubic units}$$

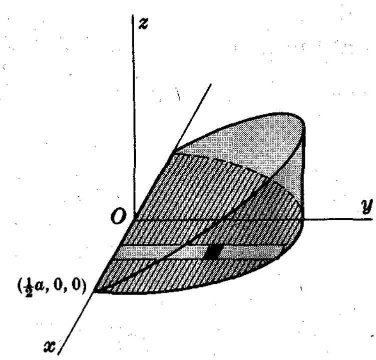

Fig. 71-4

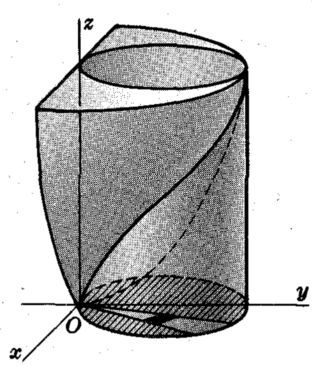

Fig. 71-5

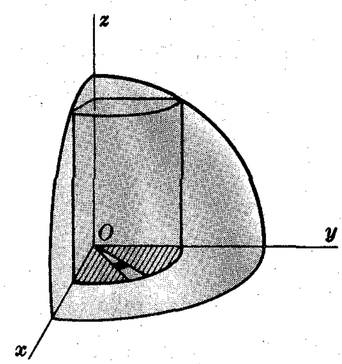

Fig. 71-6

6. Find the volume removed when a hole of radius a is bored through a sphere of radius $2a$, the axis of the hole being a diameter of the sphere. (See Fig. 71-6.)

From the figure, it is obvious that the required volume is eight times the volume in the first octant bounded by the cylinder $\rho^2 = a^2$, the sphere $\rho^2 + z^2 = 4a^2$, and the plane $z = 0$. The latter volume is obtained by integrating $z = \sqrt{4a^2 - \rho^2}$ over a quadrant of the circle $\rho = a$. Hence,

$$V = 8 \int_0^{\pi/2} \int_0^a \sqrt{4a^2 - \rho^2}\rho\, d\rho\, d\theta = \tfrac{8}{3} \int_0^{\pi/2} (8a^3 - 3\sqrt{3}a^3)\, d\theta = \tfrac{4}{3}(8 - 3\sqrt{3})a^3\pi \text{ cubic units}$$

Supplementary Problems

7. Find the volume cut from $9x^2 + 4y^2 + 36z = 36$ by the plane $z = 0$. *Ans.* 3π cubic units

8. Find the volume under $z = 3x$ and above the first-quadrant area bounded by $x = 0$, $y = 0$, $x = 4$, and $x^2 + y^2 = 25$. *Ans.* 98 cubic units

9. Find the volume in the first octant bounded by $x^2 + z = 9$, $3x + 4y = 24$, $x = 0$, $y = 0$, and $z = 0$. *Ans.* 1485/16 cubic units

10. Find the volume in the first octant bounded by $xy = 4z$, $y = x$, and $x = 4$. *Ans.* 8 cubic units

11. Find the volume in the first octant bounded by $x^2 + y^2 = 25$ and $z = y$. *Ans.* $\frac{125}{3}$ cubic units

12. Find the volume common to the cylinders $x^2 + y^2 = 16$ and $x^2 + z^2 = 16$. *Ans.* $\frac{1024}{3}$ cubic units

13. Find the volume in the first octant inside $y^2 + z^2 = 9$ and outside $y^2 = 3x$. *Ans.* $27\pi/16$ cubic units

14. Find the volume in the first octant bounded by $x^2 + z^2 = 16$ and $x - y = 0$. *Ans.* $\frac{64}{3}$ cubic units

15. Find the volume in front of $x = 0$ and common to $y^2 + z^2 = 4$ and $y^2 + z^2 + 2x = 16$. *Ans.* 28π cubic units

16. Find the volume inside $\rho = 2$ and outside the cone $z^2 = \rho^2$. *Ans.* $32\pi/3$ cubic units

17. Find the volume inside $y^2 + z^2 = 2$ and outside $x^2 - y^2 - z^2 = 2$. *Ans.* $8\pi(4 - \sqrt{2})/3$ cubic units

18. Find the volume common to $\rho^2 + z^2 = a^2$ and $\rho = a \sin\theta$. *Ans.* $2(3\pi - 4)a^2/9$ cubic units

19. Find the volume inside $x^2 + y^2 = 9$, bounded below by $x^2 + y^2 + 4z = 16$ and above by $z = 4$. *Ans.* $81\pi/8$ cubic units

20. Find the volume cut from the paraboloid $4x^2 + y^2 = 4z$ by the plane $z - y = 2$. *Ans.* 9π cubic units

21. Find the volume generated by revolving the cardioid $\rho = 2(1 - \cos\theta)$ about the polar axis. *Ans.* $V = 2\pi \int\int y\rho \, d\rho \, d\theta = 64\pi/3$ cubic units

22. Find the volume generated by revolving a petal of $\rho = \sin 2\theta$ about either axis. *Ans.* $32\pi/105$ cubic units

23. A square hole 2 units on a side is cut symmetrically through a sphere of radius 2 units. Show that the volume removed is $\frac{4}{3}(2\sqrt{2} + 19\pi - 54 \text{ Arctan} \sqrt{2})$ cubic units.

Chapter 72

Area of a Curved Surface by Double Integration

TO COMPUTE THE LENGTH OF AN ARC, (1) the arc is projected on a convenient coordinate axis, thus establishing an interval on the axis, and (2) an integrand function, $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ if the projection is on the x axis or $\sqrt{1+\left(\frac{dx}{dy}\right)^2}$ if the projection is on the y axis, is integrated over the interval.

A similar procedure is used to compute the area S of a portion R^* of a surface $z = f(x, y)$: (1) R^* is projected on a convenient coordinate plane, thus establishing a region R on the plane, and (2) an integrand function is integrated over R. Then,

If R^* is projected on xOy, $S = \displaystyle\iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA.$

If R^* is projected on yOz, $S = \displaystyle\iint_R \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\, dA.$

If R^* is projected on zOx, $S = \displaystyle\iint_R \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\, dA.$

Solved Problems

1. Derive the first of the formulas for the area S of a region R^* as given above.

Consider a region R^* of area S on the surface $z = f(x, y)$. Through the boundary of R^* pass a vertical cylinder (see Fig. 72-1) cutting the xOy plane in the region R. Now divide R into n subregions

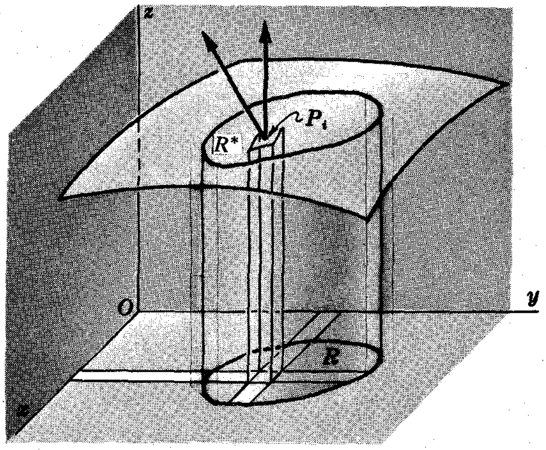

Fig. 72-1

ΔA_i (of areas ΔA_i), and denote by ΔS_i the area of the projection of ΔA_i on R^*. In each subregion ΔS_i, choose a point P_i and draw there the tangent plane to the surface. Let the area of the projection of ΔA_i on this tangent plane be denoted by ΔT_i. We shall use ΔT_i as an approximation of the corresponding surface area ΔS_i.

Now the angle between the xOy plane and the tangent plane at P_i is the angle γ_i between the z axis with direction numbers $[0, 0, 1]$, and the normal, $\left[-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right] = \left[-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right]$, to the surface at P_i; thus

$$\cos \gamma_i = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

Then (see Fig. 72-2),

$$\Delta T_i \cos \gamma_i = \Delta A_i \qquad \text{and} \qquad \Delta T_i = \sec \gamma_i \, \Delta A_i$$

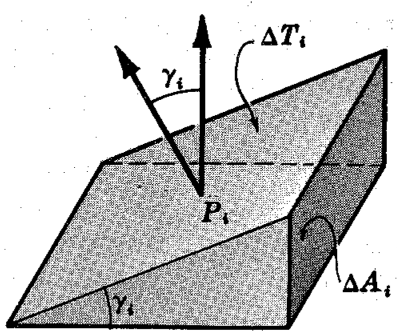

Fig. 72-2

Hence, an approximation of S is $\sum_{i=1}^{n} \Delta T_i = \sum_{i=1}^{n} \sec \gamma_i \, \Delta A_i$, and

$$S = \lim_{n \to +\infty} \sum_{i=1}^{n} \sec \gamma_i \, \Delta A_i = \iint_R \sec \gamma \, dA = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$

2. **Find the area of the portion of the cone $x^2 + y^2 = 3z^2$ lying above the xOy plane and inside the cylinder $x^2 + y^2 = 4y$.**

Solution 1: Refer to Fig. 72-3. The projection of the required area on the xOy plane is the region R enclosed by the circle $x^2 + y^2 = 4y$. For the cone,

$$\frac{\partial z}{\partial x} = \frac{1}{3}\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{3}\frac{y}{z}. \quad \text{So} \quad 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{9z^2 + x^2 + y^2}{9z^2} = \frac{12z^2}{9z^2} = \frac{4}{3}$$

Then $$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA = \int_0^4 \int_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} \frac{2}{\sqrt{3}} \, dx \, dy = 2\,\frac{2}{\sqrt{3}} \int_0^4 \int_0^{\sqrt{4y-y^2}} dx \, dy$$

$$= \frac{4}{\sqrt{3}} \int_0^4 \sqrt{4y - y^2} \, dy = \frac{8\sqrt{3}}{3}\pi \text{ square units}$$

Solution 2: Refer to Fig. 72-4. The projection of one-half the required area on the yOz plane is the region R bounded by the line $y = \sqrt{3}z$ and the parabola $y = \frac{3}{4}z^2$, the latter obtained by eliminating x between the equations of the two surfaces. For the cone,

$$\frac{\partial x}{\partial y} = -\frac{y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = \frac{3z}{x}. \quad \text{So} \quad 1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2 + y^2 + 9z^2}{x^2} = \frac{12z^2}{x^2} = \frac{12z^2}{3z^2 - y^2}$$

Then $$S = 2\int_0^4 \int_{y/\sqrt{3}}^{2\sqrt{y}/\sqrt{3}} \frac{2\sqrt{3}z}{\sqrt{3z^2 - y^2}} \, dz \, dy = \frac{4\sqrt{3}}{3} \int_0^4 [\sqrt{3z^2 - y^2}]_{y/\sqrt{3}}^{2\sqrt{y}/\sqrt{3}} \, dy = \frac{4\sqrt{3}}{3} \int_0^4 \sqrt{4y - y^2} \, dy$$

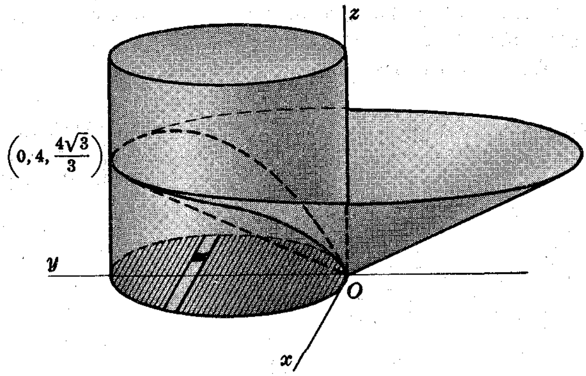

Fig. 72-3

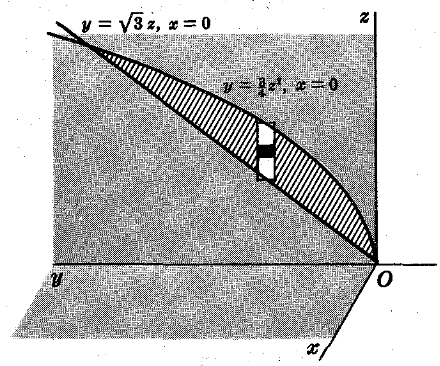

Fig. 72-4

Solution 3: Using polar coordinates in solution 1, we must integrate $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \frac{2}{\sqrt{3}}$ over the region R enclosed by the circle $\rho = 4 \sin \theta$. Then,

$$S = \iint_R \frac{2}{\sqrt{3}}\, dA = \int_0^{\pi} \int_0^{4 \sin \theta} \frac{2}{\sqrt{3}}\, \rho\, d\rho\, d\theta = \frac{1}{\sqrt{3}} \int_0^{\pi} [\rho^2]_0^{4 \sin \theta}\, d\theta$$

$$= \frac{16}{\sqrt{3}} \int_0^{\pi} \sin^2 \theta\, d\theta = \frac{8\sqrt{3}}{3}\, \pi \text{ square units}$$

3. **Find the area of the portion of the cylinder $x^2 + z^2 = 16$ lying inside the cylinder $x^2 + y^2 = 16$.**

Figure 72-5 shows one-eighth of the required area, its projection on the xOy plane being a quadrant of the circle $x^2 + y^2 = 16$. For the cylinder $x^2 + z^2 = 16$,

$$\frac{\partial z}{\partial x} = -\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y} = 0. \quad \text{So} \quad 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{x^2 + z^2}{z^2} = \frac{16}{16 - x^2}$$

Then
$$S = 8 \int_0^4 \int_0^{\sqrt{16-x^2}} \frac{4}{\sqrt{16 - x^2}}\, dy\, dx = 32 \int_0^4 dx = 128 \text{ square units}$$

4. **Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 16$ outside the paraboloid $x^2 + y^2 + z = 16$.**

Figure 72-6 shows one-fourth of the required area, its projection on the yOz plane being the region R bounded by the circle $y^2 + z^2 = 16$, the y and z axes, and the line $z = 1$. For the sphere,

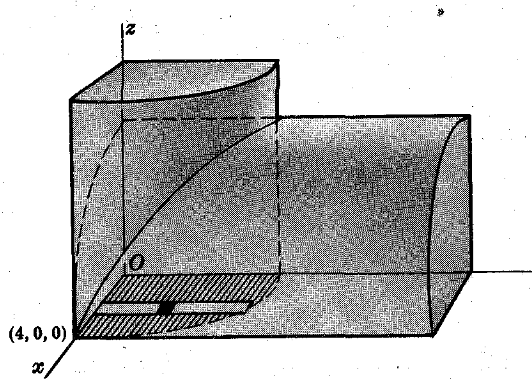

Fig. 72-5

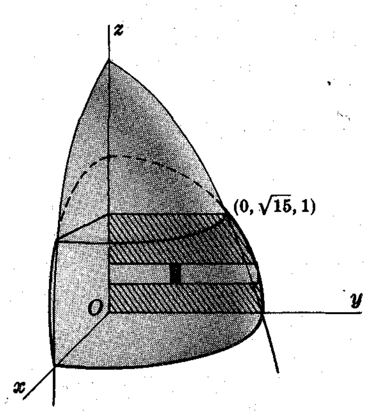

Fig. 72-6

$$\frac{\partial x}{\partial y} = -\frac{y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = -\frac{z}{x}. \quad \text{So} \quad 1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2 + y^2 + z^2}{x^2} = \frac{16}{16 - y^2 - z^2}$$

Then
$$S = 4 \iint_R \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dA = 4 \int_0^1 \int_0^{\sqrt{16-z^2}} \frac{4}{\sqrt{16 - y^2 - z^2}}\, dy\, dz$$
$$= 16 \int_0^1 \left[\arcsin \frac{y}{\sqrt{16 - z^2}}\right]_0^{\sqrt{16-z^2}} dz = 16 \int_0^1 \frac{1}{2}\pi\, dz = 8\pi \text{ square units}$$

5. Find the area of the portion of the cylinder $x^2 + y^2 = 6y$ lying inside the sphere $x^2 + y^2 + z^2 = 36$.

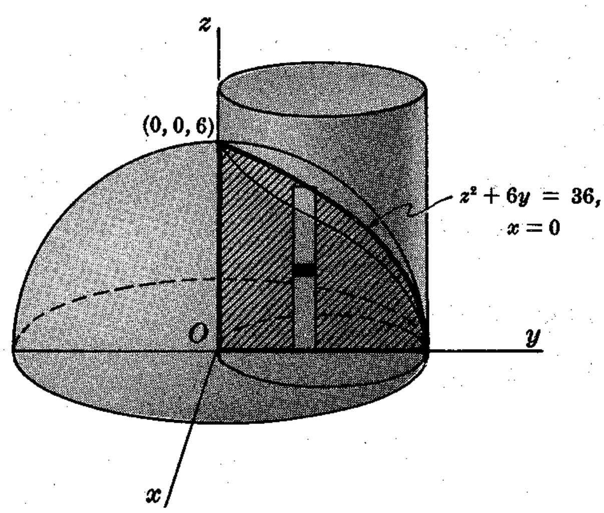

Fig. 72-7

Figure 72-7 shows one-fourth of the required area. Its projection on the yOz plane is the region R bounded by the z and y axes and the parabola $z^2 + 6y = 36$, the latter obtained by eliminating x from the equations of the two surfaces. For the cylinder,

$$\frac{\partial x}{\partial y} = \frac{3 - y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = 0. \quad \text{So} \quad 1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2 + 9 - 6y + y^2}{x^2} = \frac{9}{6y - y^2}$$

Then
$$S = 4 \int_0^6 \int_0^{\sqrt{36-6y}} \frac{3}{\sqrt{6y - y^2}}\, dz\, dy = 12 \int_0^6 \frac{\sqrt{6}}{\sqrt{y}}\, dy = 144 \text{ square units}$$

Supplementary Problems

6. Find the area of the portion of the cone $x^2 + y^2 = z^2$ inside the vertical prism whose base is the triangle bounded by the lines $y = x$, $x = 0$, and $y = 1$ in the xOy plane. *Ans.* $\frac{1}{2}\sqrt{2}$ square units

7. Find the area of the portion of the plane $x + y + z = 6$ inside the cylinder $x^2 + y^2 = 4$.
Ans. $4\sqrt{3}\pi$ square units

8. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 36$ inside the cylinder $x^2 + y^2 = 6y$.
Ans. $72(\pi - 2)$ square units

9. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 4z$ inside the paraboloid $x^2 + y^2 = z$.

Ans. 4π square units

10. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 25$ between the planes $z = 2$ and $z = 4$.

Ans. 20π square units

11. Find the area of the portion of the surface $z = xy$ inside the cylinder $x^2 + y^2 = 1$.

Ans. $2\pi(2\sqrt{2} - 1)/3$ square units

12. Find the area of the surface of the cone $x^2 + y^2 - 9z^2 = 0$ above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 6y$. *Ans.* $3\sqrt{10}\pi$ square units

13. Find the area of that part of the sphere $x^2 + y^2 + z^2 = 25$ that is within the elliptic cylinder $2x^2 + y^2 = 25$.

Ans. 50π square units

14. Find the area of the surface of $x^2 + y^2 - az = 0$ which lies directly above the lemniscate $4\rho^2 = a^2 \cos 2\theta$. *Ans.* $S = \frac{4}{a} \int\int \sqrt{4\rho^2 + a^2}\rho \, d\rho \, d\theta = \frac{a^2}{3}\left(\frac{5}{3} - \frac{\pi}{4}\right)$ square units

15. Find the area of the surface of $x^2 + y^2 + z^2 = 4$ which lies directly above the cardioid $\rho = 1 - \cos\theta$.

Ans. $8[\pi - \sqrt{2} - \ln(\sqrt{2} + 1)]$ square units

Chapter 73

Triple Integrals

CYLINDRICAL AND SPHERICAL COORDINATES. Assume that a point P has coordinates (x, y, z) in a right-handed rectangular coordinate system. The corresponding *cylindrical coordinates* of P are (r, θ, z), where (r, θ) are the polar coordinates for the point (x, y) in the xy plane. (Note the notational change here from (ρ, θ) to (r, θ) for the polar coordinates of (x, y); see Fig. 73-1.) Hence we have the relations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

In cylindrical coordinates, an equation $r = c$ represents a right circular cylinder of radius c with the z axis as its axis of symmetry. An equation $\theta = c$ represents a plane through the z axis.

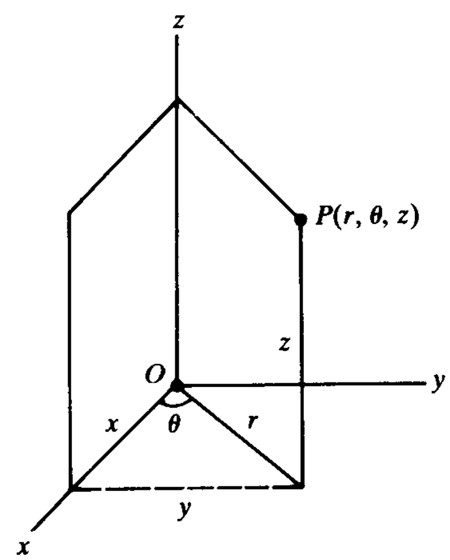

Fig. 73-1

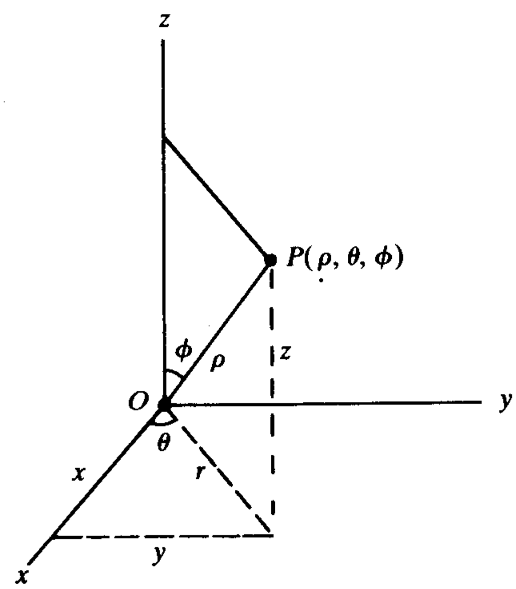

Fig. 73-2

A point P with rectangular coordinates (x, y, z) has the *spherical coordinates* (ρ, θ, ϕ), where $\rho = |OP|$, θ is the same as in cylindrical coordinates, and ϕ is the directed angle from the positive z axis to the vector **OP**. (See Fig. 73-2.) In spherical coordinates, an equation $\rho = c$ represents a sphere of radius c with center at the origin. An equation $\phi = c$ represents a cone with vertex at the origin and the z axis as its axis of symmetry.

The following additional relations hold among spherical, cylindrical, and rectangular coordinates:

$$r = \rho\sin\phi \qquad z = \rho\cos\phi \qquad \rho^2 = x^2 + y^2 + z^2$$
$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta$$

(See Problems 14 to 16.)

THE TRIPLE INTEGRAL $\iiint\limits_R f(x, y, z)\,dV$ of a function of three independent variables over a closed region R of points (x, y, z), of volume V, on which the function is single-valued and continuous, is an extension of the notion of single and double integrals.

If $f(x, y, z) = 1$, then $\iiint\limits_R f(x, y, z)\, dV$ may be interpreted as measuring the volume of the region R.

EVALUATION OF THE TRIPLE INTEGRAL. In rectangular coordinates,

$$\iiint\limits_R f(x, y, z)\, dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

$$= \int_c^d \int_{x_1(y)}^{x_2(y)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dx\, dy, \text{ etc.}$$

where the limits of integration are chosen to cover the region R.

In cylindrical coordinates,

$$\iiint\limits_R f(r, \theta, z)\, dV = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r, \theta, z) r\, dz\, dr\, d\theta$$

where the limits of integration are chosen to cover the region R.

In spherical coordinates,

$$\iiint\limits_R f(\rho, \phi, \theta)\, dV = \int_\alpha^\beta \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho, \phi, \theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

where the limits of integration are chosen to cover the region R.

Discussion of the definitions: Consider the function $f(x, y, z)$, continuous over a region R of ordinary space. After slicing R with planes $x = \xi_i$ and $y = \eta_j$ as in Chapter 69, let these subregions be further sliced by planes $z = \zeta_k$. The region R has now been separated into a number of rectangular parallelepipeds of volume $\Delta V_{ijk} = \Delta x_i\, \Delta y_j\, \Delta z_k$ and a number of partial parallelepipeds which we shall ignore. In each complete parallelepiped select a point $P_{ijk}(x_i, y_j, z_k)$; then compute $f(x_i, y_j, z_k)$ and form the sum

$$\sum_{\substack{i=1,\dots,m \\ j=1,\dots,n \\ k=1,\dots,p}} f(x_i, y_j, z_k)\, \Delta V_{ijk} = \sum_{\substack{i=1,\dots,m \\ j=1,\dots,n \\ k=1,\dots,p}} f(x_i, y_j, z_k)\, \Delta x_i\, \Delta y_j\, \Delta z_k \tag{73.1}$$

The triple integral of $f(x, y, z)$ over the region R is defined to be the limit of (*73.1*) as the number of parallelepipeds is indefinitely increased in such a manner that all dimensions of each go to zero.

In evaluating this limit, we may sum first each set of parallelepipeds having $\Delta_i x$ and $\Delta_j y$, for fixed i and j, as two dimensions and consider the limit as each $\Delta_k z \to 0$. We have

$$\lim_{p\to+\infty} \sum_{k=1}^{p} f(x_i, y_j, z_k)\, \Delta_k z\, \Delta_i x\, \Delta_j y = \int_{z_1}^{z_2} f(x_i, y_i, z)\, dz\, \Delta_i x\, \Delta_j y$$

Now these are the columns, the basic subregions, of Chapter 69; hence,

$$\lim_{\substack{m\to+\infty \\ n\to+\infty \\ p\to+\infty}} \sum_{\substack{i=1,\dots,m \\ j=1,\dots,n \\ k=1,\dots,p}} f(x_i, y_j, z_k)\, \Delta V_{ijk} = \iiint\limits_R f(x, y, z)\, dz\, dx\, dy = \iiint\limits_R f(x, y, z)\, dz\, dy\, dx$$

CENTROIDS AND MOMENTS OF INERTIA. The coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the *centroid of a volume* satisfy the relations

$$\bar{x}\iiint\limits_R dV=\iiint\limits_R x\,dV \qquad \bar{y}\iiint\limits_R dV=\iiint\limits_R y\,dV$$

$$\bar{z}\iiint\limits_R dV=\iiint\limits_R z\,dV$$

The *moments of inertia of a volume* with respect to the coordinate axes are given by

$$I_x=\iiint\limits_R (y^2+z^2)\,dV \qquad I_y=\iiint\limits_R (z^2+x^2)\,dV \qquad I_z=\iiint\limits_R (x^2+y^2)\,dV$$

Solved Problems

1. Evaluate the given triple integrals:

(*a*) $\int_0^1 \int_0^{1-x} \int_0^{2-x} xyz\,dz\,dy\,dx$

$$=\int_0^1\left[\int_0^{1-x}\left(\int_0^{2-x} xyz\,dz\right)dy\right]dx$$

$$=\int_0^1\left[\int_0^{1-x}\left(\frac{xyz^2}{2}\bigg|_{z=0}^{z=2-x}\right)dy\right]dx=\int_0^1\left[\int_0^{1-x}\frac{xy(2-x)^2}{2}\,dy\right]dx$$

$$=\int_0^1\left[\frac{xy^2(2-x)^2}{4}\right]_{y=0}^{y=1-x}dx=\frac{1}{4}\int_0^1(4x-12x^2+13x^3-6x^4+x^5)\,dx=\frac{13}{240}$$

(*b*) $\int_0^{\pi/2}\int_0^1\int_0^2 zr^2\sin\theta\,dz\,dr\,d\theta$

$$=\int_0^{\pi/2}\int_0^1\left[\frac{z^2}{2}\right]_0^2 r^2\sin\theta\,dr\,d\theta=2\int_0^{\pi/2}\int_0^1 r^2\sin\theta\,dr\,d\theta$$

$$=\frac{2}{3}\int_0^{\pi/2}[r^3]_0^1\sin\theta\,d\theta=-\frac{2}{3}[\cos\theta]_0^{\pi/2}=\frac{2}{3}$$

(*c*) $\int_0^{\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\sin 2\phi\,d\rho\,d\phi\,d\theta$

$$=2\int_0^{\pi}\int_0^{\pi/4}\sin\phi\,d\phi\,d\theta=2\int_0^{\pi}(1-\tfrac{1}{2}\sqrt{2})\,d\theta=(2-\sqrt{2})\pi$$

2. Compute the triple integral of $F(x, y, z) = z$ over the region R in the first octant bounded by the planes $y=0$, $z=0$, $x+y=2$, $2y+x=6$, and the cylinder $y^2+z^2=4$. (See Fig. 73-3.)

Integrate first with respect to z from $z=0$ (the xOy plane) to $z=\sqrt{4-y^2}$ (the cylinder), then with respect to x from $x=2-y$ to $x=6-2y$, and finally with respect to y from $y=0$ to $y=2$. This yields

$$\iiint\limits_R z\,dV=\int_0^2\int_{2-y}^{6-2y}\int_0^{\sqrt{4-y^2}} z\,dz\,dx\,dy=\int_0^2\int_{2-y}^{6-2y}[\tfrac{1}{2}z^2]_0^{\sqrt{4-y^2}}\,dx\,dy$$

$$=\tfrac{1}{2}\int_0^2\int_{2-y}^{6-2y}(4-y^2)\,dx\,dy=\tfrac{1}{2}\int_0^2[(4-y^2)x]_{2-y}^{6-2y}\,dy=\tfrac{26}{3}$$

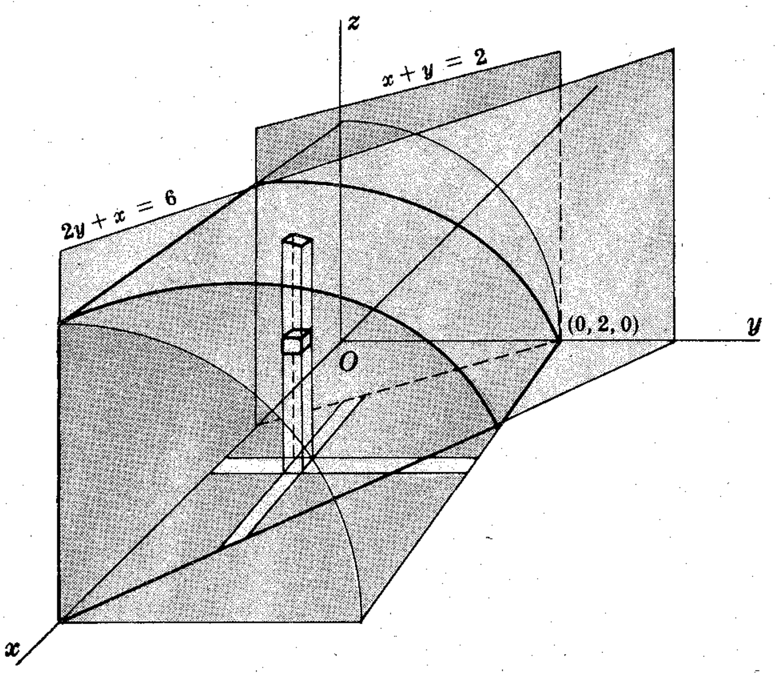

Fig. 73-3

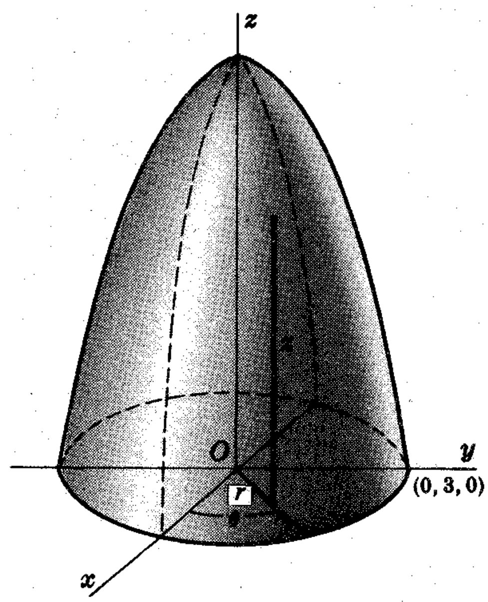

Fig. 73-4

3. Compute the triple integral of $f(r, \theta, z) = r^2$ over the region R bounded by the paraboloid $r^2 = 9 - z$ and the plane $z = 0$. (See Fig. 73-4.)

Integrate first with respect to z from $z = 0$ to $z = 9 - r^2$, then with respect to r from $r = 0$ to $r = 3$, and finally with respect to θ from $\theta = 0$ to $\theta = 2\pi$. This yields

$$\iiint_R r^2\, dV = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2(r\, dz\, dr\, d\theta) = \int_0^{2\pi}\int_0^3 r^3(9 - r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{9}{4}r^4 - \tfrac{1}{6}r^6\right]_0^3 d\theta = \int_0^{2\pi} \tfrac{243}{4}\, d\theta = \tfrac{243}{2}\pi$$

4. Show that the integrals (a) $4\int_0^4\int_0^{\sqrt{16-x^2}}\int_{(x^2+y^2)/4}^4 dz\, dy\, dx$, (b) $4\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$, and (c) $4\int_0^4\int_{y^2/4}^4\int_0^{\sqrt{4z-y^2}} dx\, dz\, dy$ give the same volume.

(a) Here z ranges from $z = \frac{1}{4}(x^2 + y^2)$ to $z = 4$; that is, the volume is bounded below by the paraboloid $4z = x^2 + y^2$ and above the plane $z = 4$. The ranges of y and x cover a quadrant of the circle $x^2 + y^2 = 16$, $z = 0$, the projection of the curve of intersection of the paraboloid and the plane $z = 4$ on the xOy plane. Thus, the integral gives the volume cut from the paraboloid by the plane $z = 4$.

(b) Here y ranges from $y = 0$ to $y = \sqrt{4z - x^2}$; that is, the volume is bounded on the left by the zOx plane and on the right by the paraboloid $y^2 = 4z - x^2$. The ranges of x and z cover one-half the area cut from the parabola $x^2 = 4z$, $y = 0$, the curve of intersection of the paraboloid and the zOx plane, by the plane $z = 4$. The region R is that of (a).

(c) Here the volume is bounded behind by the yOz plane and in front by the paraboloid $4z = x^2 + y^2$. The ranges of z and y cover one-half the area cut from the parabola $y^2 = 4z$, $x = 0$, the curve of intersection of the paraboloid and the yOz plane, by the plane $z = 4$. The region R is that of (a).

5. Compute the triple integral of $F(\rho, \phi, \theta) = 1/\rho$ over the region R in the first octant bounded by the cones $\phi = \frac{1}{4}\pi$ and $\phi = \arctan 2$ and the sphere $\rho = \sqrt{6}$. (See Fig. 73-5.)

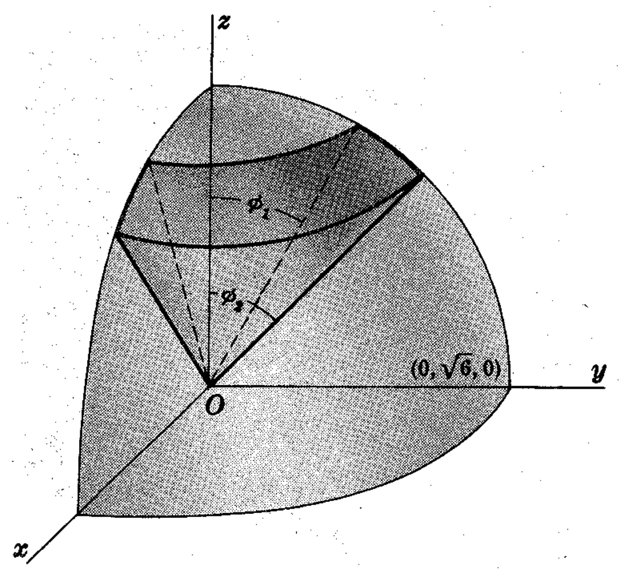

Fig. 73-5

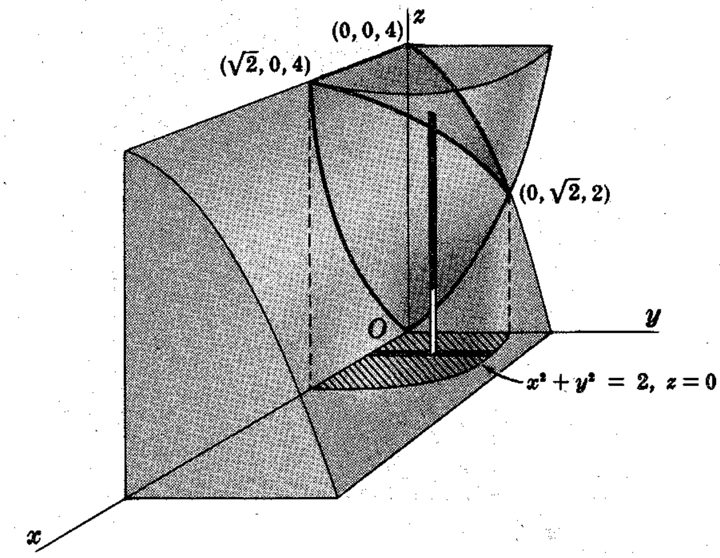

Fig. 73-6

Integrate first with respect to ρ from $\rho = 0$ to $\rho = \sqrt{6}$, then with respect to ϕ from $\phi = \frac{1}{4}\pi$ to $\phi = \arctan 2$, and finally with respect to θ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. This yields

$$\iiint_R \frac{1}{\rho}\,dV = \int_0^{\pi/2}\int_{\pi/4}^{\arctan 2}\int_0^{\sqrt{6}} \frac{1}{\rho}\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = 3\int_0^{\pi/2}\int_{\pi/4}^{\arctan 2} \sin\phi\,d\phi\,d\theta$$

$$= -3\int_0^{\pi/2}\left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{2}}\right)d\theta = \frac{3\pi}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}}\right)$$

6. Find the volume bounded by the paraboloid $z = 2x^2 + y^2$ and the cylinder $z = 4 - y^2$. (See Fig. 73-6.)

Integrate first with respect to z from $z = 2x^2 + y^2$ to $z = 4 - y^2$, then with respect to y from $y = 0$ to $y = \sqrt{2 - x^2}$ (obtain $x^2 + y^2 = 2$ by eliminating x between the equations of the two surfaces), and finally with respect to x from $x = 0$ to $x = \sqrt{2}$ (obtained by setting $y = 0$ in $x^2 + y^2 = 2$) to obtain one-fourth of the required volume. Thus,

$$V = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} dz\,dy\,dx = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} [(4-y^2) + (2x^2+y^2)]\,dy\,dx$$

$$= 4\int_0^{\sqrt{2}}\left[4y - 2x^2y - \frac{2y^3}{3}\right]_0^{\sqrt{2-x^2}} dx = \frac{16}{3}\int_0^{\sqrt{2}} (2-x^2)^{3/2}\,dx = 4\pi \text{ cubic units}$$

7. Find the volume within the cylinder $r = 4\cos\theta$ bounded above by the sphere $r^2 + z^2 = 16$ and below by the plane $z = 0$. (See Fig. 73-7.)

Integrate first with respect to z from $z = 0$ to $z = \sqrt{16 - r^2}$, then with respect to r from $r = 0$ to $r = 4\cos\theta$, and finally with respect to θ from $\theta = 0$ to $\theta = \pi$ to obtain the required volume. Thus,

$$V = \int_0^{\pi}\int_0^{4\cos\theta}\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{4\cos\theta} r\sqrt{16 - r^2}\,dr\,d\theta$$

$$= -\tfrac{64}{3}\int_0^{\pi} (\sin^3\theta - 1)\,d\theta = \tfrac{64}{9}(3\pi - 4) \text{ cubic units}$$

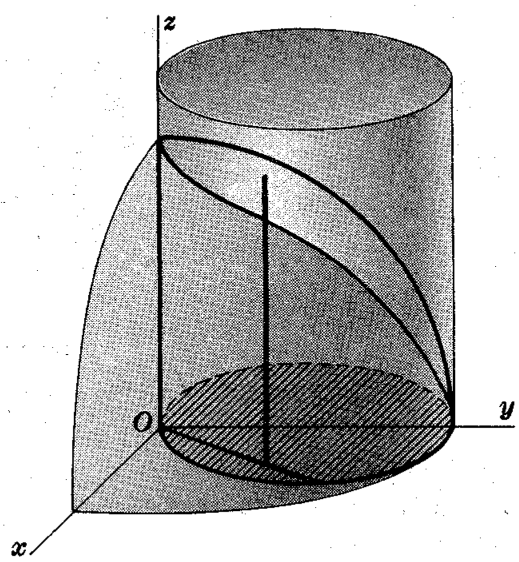

Fig. 73-7

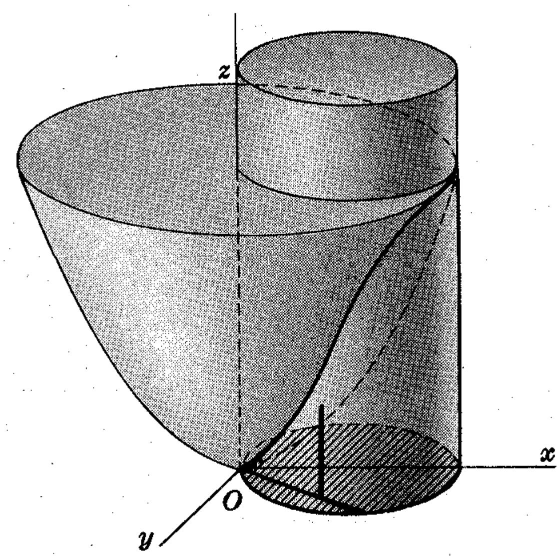

Fig. 73-8

8. Find the coordinates of the centroid of the volume within the cylinder $r = 2\cos\theta$, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$. (See Fig. 73-8.)

$$V = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi/2}[r^4]_0^{2\cos\theta}\,d\theta = 8\int_0^{\pi/2}\cos^4\theta\,d\theta = \tfrac{3}{2}\pi$$

$$M_{yz} = \iiint_R x\,dV = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2}(r\cos\theta)r\,dz\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^4\cos\theta\,dr\,d\theta = \tfrac{64}{5}\int_0^{\pi/2}\cos^6\theta\,d\theta = 2\pi$$

Then $\bar{x} = M_{yz}/V = \frac{4}{3}$. By symmetry, $\bar{y} = 0$. Also,

$$M_{xy} = \iiint_R z\,dV = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} zr\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta} r^5\,dr\,d\theta$$

$$= \tfrac{32}{3}\int_0^{\pi/2}\cos^6\theta\,d\theta = \tfrac{5}{3}\pi$$

and $\bar{z} = M_{xy}/V = \frac{10}{9}$. Thus, the centroid has coordinates $(\frac{4}{3}, 0, \frac{10}{9})$.

9. For the right circular cone of radius a and height h, find (a) the centroid, (b) the moment of inertia with respect to its axis (c), the moment of inertia with respect to any line through its vertex and perpendicular to its axis, (d) the moment of inertia with respect to any line through its centroid and perpendicular to its axis, an (e) the moment of inertia with respect to any diameter of its base.

Take the cone as in Fig. 73-9, so that its equation is $r = \dfrac{a}{h}z$. Then

$$V = 4\int_0^{\pi/2}\int_0^{a}\int_{hr/a}^{h} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{a}\left(hr - \frac{h}{a}r^2\right)dr\,d\theta$$

$$= \frac{2}{3}ha^2\int_0^{\pi/2} d\theta = \frac{1}{3}\pi ha^2$$

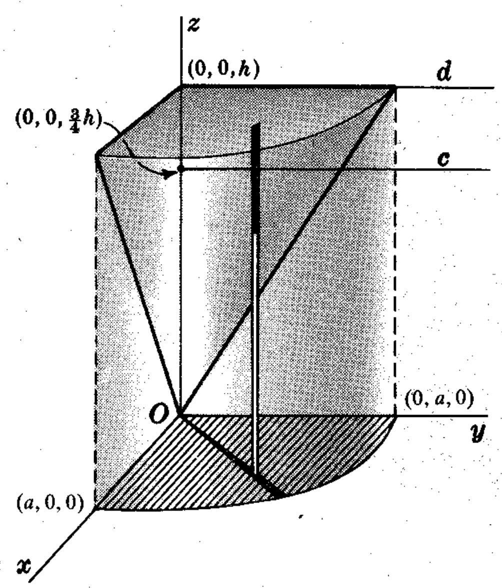

Fig. 73-9

(*a*) The centroid lies on the z axis, and we have

$$M_{xy} = \iiint_R z\,dV = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h zr\,dz\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^a \left(h^2r - \frac{h^2}{a^2}r^3\right) dr\,d\theta = \frac{1}{2}h^2a^2\int_0^{\pi/2} d\theta = \frac{1}{4}\pi h^2a^2$$

Then $\bar{z} = M_{xy}/V = \frac{3}{4}h$, and the centroid has coordinates $(0, 0, \frac{3}{4}h)$.

(*b*) $$I_z = \iiint_R (x^2+y^2)\,dV = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h (r^2)r\,dz\,dr\,d\theta = \tfrac{1}{10}\pi ha^4 = \tfrac{3}{10}a^2V$$

(*c*) Take the line as the y axis. Then

$$I_y = \iiint_R (x^2+z^2)\,dV = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h (r^2\cos^2\theta + z^2)r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^a \left[\left(hr^3 - \frac{h}{a}r^4\right)\cos^2\theta + \frac{1}{3}\left(h^3r - \frac{h^3}{a^3}r^4\right)\right] dr\,d\theta$$

$$= \frac{1}{5}\pi ha^2\left(h^2 + \frac{1}{4}a^2\right) = \frac{3}{5}\left(h^2 + \frac{1}{4}a^2\right)V$$

(*d*) Let the line c through the centroid be parallel to the y axis. By the parallel-axis theorem,

$$I_y = I_c + V(\tfrac{3}{4}h)^2 \qquad \text{and} \qquad I_c = \tfrac{3}{5}(h^2 + \tfrac{1}{4}a^2)V - \tfrac{9}{16}h^2V = \tfrac{3}{80}(h^2+4a^2)V$$

(*e*) Let d denote the diameter of the base of the cone parallel to the y axis. Then

$$I_d = I_c + V(\tfrac{1}{4}h)^2 = \tfrac{3}{80}(h^2+4a^2)V + \tfrac{1}{16}h^2V = \tfrac{1}{20}(2h^2+3a^2)V$$

10. Find the volume cut from the cone $\phi = \frac{1}{4}\pi$ by the sphere $\rho = 2a\cos\phi$. (See Fig. 73-10.)

$$V = 4\iiint_R dV = 4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2a\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{32a^3}{3}\int_0^{\pi/2}\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi\,d\theta = 2a^3\int_0^{\pi/2} d\theta = \pi a^3 \text{ cubic units}$$

11. Locate the centroid of the volume cut from one nappe of a cone of vertex angle 60° by a sphere of radius 2 whose center is at the vertex of the cone.

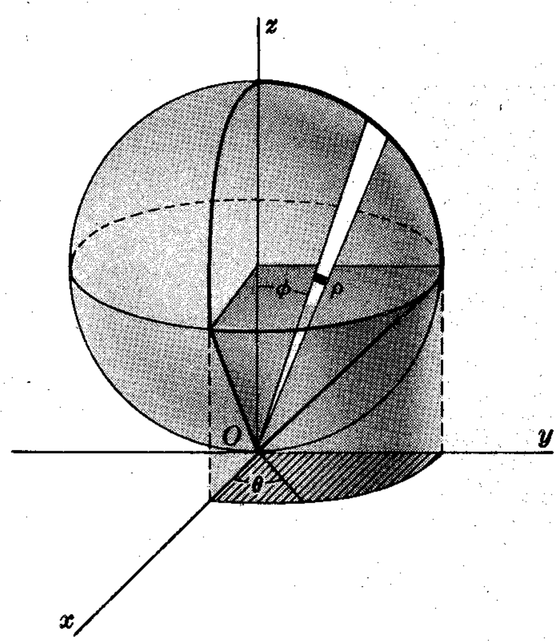

Fig. 73-10

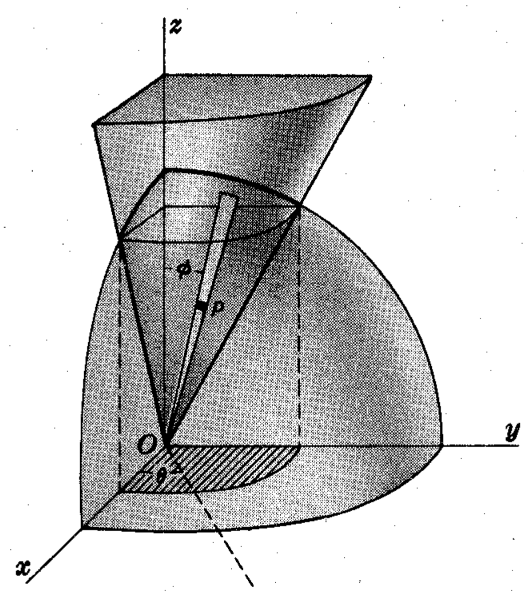

Fig. 73-11

Take the surfaces as in Fig. 73-11, so that $\bar{x} = \bar{y} = 0$. In spherical coordinates, the equation of the cone is $\phi = \pi/6$, and the equation of the sphere is $\rho = 2$. Then

$$V = \iiint\limits_R dV = 4\int_0^{\pi/2}\int_0^{\pi/6}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{32}{3}\int_0^{\pi/2}\int_0^{\pi/6} \sin\phi \, d\phi \, d\theta$$

$$= -\frac{32}{3}\left(\frac{\sqrt{3}}{2} - 1\right)\int_0^{\pi/2} d\theta = \frac{8\pi}{3}(2 - \sqrt{3})$$

$$M_{xy} = \iiint\limits_R z \, dV = 4\int_0^{\pi/2}\int_0^{\pi/6}\int_0^2 (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/6} \sin 2\phi \, d\phi \, d\theta = \pi$$

and $\bar{z} = M_{xy}/V = \frac{3}{8}(2 + \sqrt{3})$.

12. Find the moment of inertia with respect to the z axis of the volume of Problem 11.

$$I_z = \iiint\limits_R (x^2 + y^2) \, dV = 4\int_0^{\pi/2}\int_0^{\pi/6}\int_0^2 (\rho^2 \sin^2\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{128}{5}\int_0^{\pi/2}\int_0^{\pi/6} \sin^3\phi \, d\phi \, d\theta = \frac{128}{5}\left(\frac{2}{3} - \frac{3}{8}\sqrt{3}\right)\int_0^{\pi/2} d\theta = \frac{8\pi}{15}(16 - 9\sqrt{3}) = \frac{5 - 2\sqrt{3}}{5} V$$

Supplementary Problems

13. Describe the curve determined by each of the following pairs of equations in cylindrical coordinates.
(*a*) $r = 1, z = 2$ (*b*) $r = 2, z = \theta$ (*c*) $\theta = \pi/4, r = \sqrt{2}$ (*d*) $\theta = \pi/4, z = r$

Ans. (*a*) circle of radius 1 in plane $z = 2$ with center having rectangular coordinates $(0, 0, 2)$; (*b*) helix on right circular cylinder $r = 2$; (*c*) vertical line through point having rectangular coordinates $(1, 1, 0)$; (*d*) line through origin in plane $\theta = \pi/4$, making an angle of 45° with xy plane

14. Describe the curve determined by each of the following pairs of equations in spherical coordinates.

(*a*) $\rho = 1,\ \theta = \pi$ (*b*) $\theta = \frac{\pi}{4},\ \phi = \frac{\pi}{6}$ (*c*) $\rho = 2,\ \phi = \frac{\pi}{4}$

Ans. (*a*) circle of radius 1 in xz plane with center at origin; (*b*) halfline on intersection of plane $\theta = \pi/4$ and cone $\phi = \pi/6$; (*c*) circle of radius $\sqrt{2}$ in plane $z = \sqrt{2}$ with center on z axis

15. Transform each of the following equations in either rectangular, cylindrical, or spherical coordinates into equivalent equations in the two other coordinate systems.
(*a*) $\rho = 5$ (*b*) $z^2 = r^2$ (*c*) $x^2 + y^2 + (z-1)^2 = 1$

Ans. (*a*) $x^2 + y^2 + z^2 = 25$, $r^2 + z^2 = 25$; (*b*) $z^2 = x^2 + y^2$, $\cos^2\phi = \frac{1}{2}$ (that is, $\phi = \pi/4$ or $\phi = 3\pi/4$); (*c*) $r^2 + z^2 = 2z$, $\rho = 2\cos\phi$

16. Evaluate the triple integral on the left in each of the following:

(*a*) $\int_0^1 \int_1^2 \int_2^3 dz\,dx\,dy = 1$

(*b*) $\int_0^1 \int_{x^2}^{x} \int_0^{xy} dz\,dy\,dx = \frac{1}{24}$

(*c*) $\int_0^6 \int_0^{12-2y} \int_0^{4-2y/3-x/3} x\,dz\,dx\,dy = 144 \left[= \int_0^{12} \int_0^{6-x/2} \int_0^{4-2y/3-x/3} x\,dz\,dy\,dx\right]$

(*d*) $\int_0^{\pi/2} \int_0^4 \int_0^{\sqrt{16-z^2}} (16-r^2)^{1/2} rz\,dr\,dz\,d\theta = \frac{256}{5}\pi$

(*e*) $\int_0^{2\pi} \int_0^{\pi} \int_0^5 \rho^4 \sin\phi\,d\rho\,d\phi\,d\theta = 2500\pi$

17. Evaluate the integral of Problem 16(*b*) after changing the order to $dz\,dx\,dy$.

18. Evaluate the integral of Problem 16(*c*), changing the order to $dx\,dy\,dz$ and to $dy\,dz\,dx$.

19. Find the following volumes, using triple integrals in rectangular coordinates:
(*a*) Inside $x^2 + y^2 = 9$, above $z = 0$, and below $x + z = 4$ *Ans.* 36π cubic units
(*b*) Bounded by the coordinate planes and $6x + 4y + 3z = 12$ *Ans.* 4 cubic units
(*c*) Inside $x^2 + y^2 = 4x$, above $z = 0$, and below $x^2 + y^2 = 4z$ *Ans.* 6π cubic units

20. Find the following volumes, using triple integrals in cylindrical coordinates:
(*a*) The volume of Problem 4
(*b*) The volume of Problem 19(*c*)
(*c*) That inside $r^2 = 16$, above $z = 0$, and below $2z = y$ *Ans.* 64/3 cubic units

21. Find the centroid of each of the following volumes:
(*a*) Under $z^2 = xy$ and above the triangle $y = x$, $y = 0$, $x = 4$ in the plane $z = 0$ *Ans.* $(3, \frac{9}{5}, \frac{9}{8})$
(*b*) That of Problem 19(*b*) *Ans.* $(\frac{1}{2}, \frac{3}{4}, 1)$
(*c*) The first-octant volume of Problem 19(*a*) *Ans.* $\left(\frac{64-9\pi}{16(\pi-1)}, \frac{23}{8(\pi-1)}, \frac{73\pi-128}{32(\pi-1)}\right)$
(*d*) That of Problem 19(*c*) *Ans.* $(\frac{8}{3}, 0, \frac{10}{9})$
(*e*) That of Problem 20(*c*) *Ans.* $(0, 3\pi/4, 3\pi/16)$

22. Find the moments of inertia I_x, I_y, I_z of the following volumes:
(*a*) That of Problem 4 *Ans.* $I_x = I_y = \frac{32}{3}V$; $I_z = \frac{16}{3}V$
(*b*) That of Problem 19(*b*) *Ans.* $I_x = \frac{5}{2}V$; $I_y = 2V$; $I_z = \frac{13}{10}V$
(*c*) That of Problem 19(*c*) *Ans.* $I_x = \frac{55}{18}V$; $I_y = \frac{175}{18}V$; $I_z = \frac{80}{9}V$
(*d*) That cut from $z = r^2$ by the plane $z = 2$ *Ans.* $I_x = I_y = \frac{7}{3}V$; $I_z = \frac{2}{3}V$

23. Show that, in cylindrical coordinates, the triple integral of a function $f(r, \theta, z)$ over a region R may be represented by

$$\int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r, \theta, z) r \, dz \, dr \, d\theta$$

(*Hint*: Consider, in Fig. 73-12, a representative subregion of R bounded by two cylinders having Oz as axis and of radii r and $r + \Delta r$, respectively, cut by two horizontal planes through $(0, 0, z)$ and $(0, 0, z + \Delta z)$, respectively, and by two vertical planes through Oz making angles θ and $\theta + \Delta\theta$, respectively, with the xOz plane. Take $\Delta V = (r\,\Delta\theta)\,\Delta r\,\Delta z$ as an approximation of its volume.)

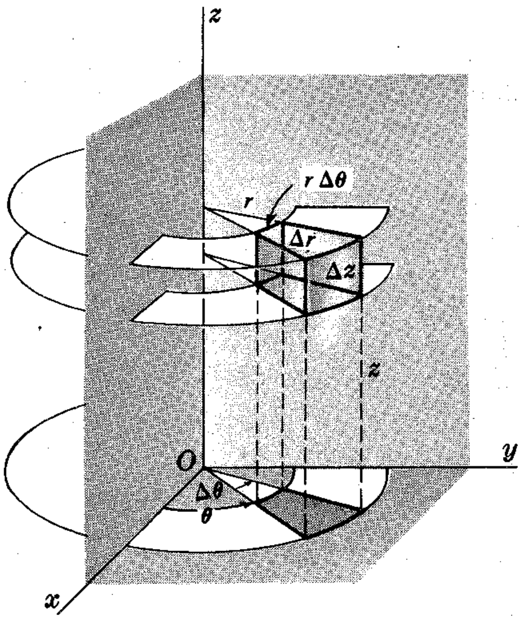

Fig. 73-12

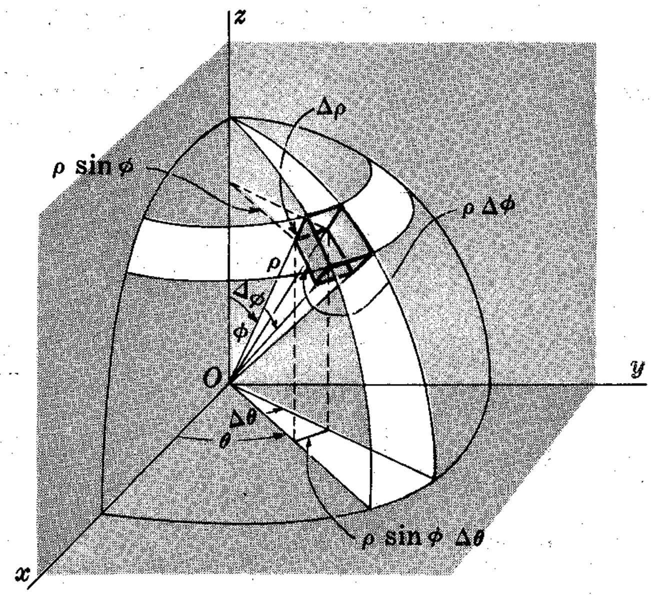

Fig. 73-13

24. Show that, in spherical coordinates, the triple integral of a function $f(\rho, \phi, \theta)$ over a region R may be represented by

$$\int_{\alpha}^{\beta} \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho, \phi, \theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(*Hint*: Consider, in Fig. 73-13, a representative subregion of R bounded by two spheres centered at O, of radii ρ and $\rho + \Delta\rho$, respectively, by two cones having O as vertex, Oz as axis, and semivertical angles ϕ and $\phi + \Delta\phi$, respectively, and by two vertical planes through Oz making angles θ and $\theta + \Delta\theta$, respectively, with the zOy plane. Take $\Delta V = (\rho\,\Delta\phi)(\rho \sin\phi\,\Delta\theta)(\Delta\rho) = \rho^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$ as an approximation of its volume.)

Chapter 74

Masses of Variable Density

HOMOGENEOUS MASSES have been treated in previous chapters as geometric figures by taking the density $\delta = 1$. The mass of a homogeneous body of volume V and density δ is $m = \delta V$.

For a nonhomogeneous mass whose density δ varies continuously from point to point, an element of mass dm is given by:

$\delta(x, y)\, ds$ for a material curve (e.g., a piece of fine wire)

$\delta(x, y)\, dA$ for a material two-dimensional plate (e.g., a thin sheet of metal)

$\delta(x, y, z)\, dV$ for a material body

Solved Problems

1. Find the mass of a semicircular wire whose density varies as the distance from the diameter joining the ends.

Take the wire as in Fig. 74-1, so that $\delta(x, y) = ky$. Then, from $x^2 + y^2 = r^2$,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \frac{r}{y}\, dx$$

and

$$m = \int \delta(x, y)\, ds = \int_{-r}^{r} ky\, \frac{r}{y}\, dx = kr \int_{-r}^{r} dx = 2kr^2 \text{ units}$$

2. Find the mass of a square plate of side a if the density varies as the square of the distance from a vertex.

Take the square as in Fig. 74-2, and let the vertex from which distances are measured be at the origin. Then $\delta(x, y) = k(x^2 + y^2)$ and

$$m = \iint_R \delta(x, y)\, dA = \int_0^a \int_0^a k(x^2 + y^2)\, dx\, dy = k \int_0^a \left(\tfrac{1}{3}a^3 + ay^2\right) dy = \tfrac{2}{3}ka^4 \text{ units}$$

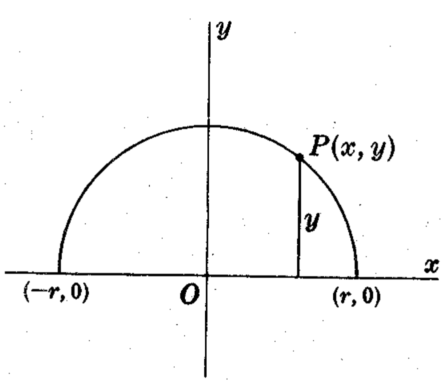

Fig. 74-1

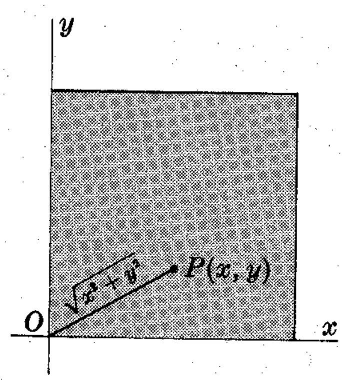

Fig. 74-2

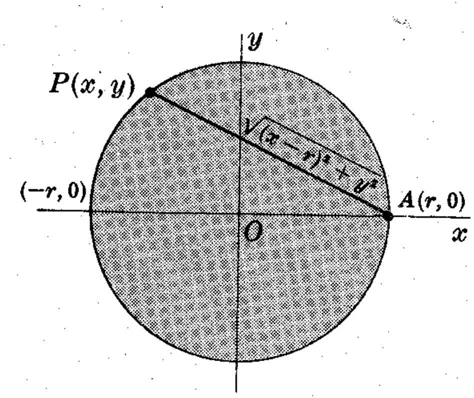

Fig. 74-3

3. Find the mass of a circular plate of radius r if the density varies as the square of the distance from a point on the circumference.

Take the circle as in Fig. 74-3, and let $A(r, 0)$ be the fixed point on the circumference. Then $\delta(x, y) = k[(x-r)^2 + y^2]$ and

$$m = \iint_R \delta(x, y)\, dA = 2\int_{-r}^{r}\int_0^{\sqrt{r^2-x^2}} k[(x-r)^2 + y^2]\, dy\, dx = \tfrac{3}{2}k\pi r^4 \text{ units}$$

4. Find the center of mass of a plate in the form of the segment cut from the parabola $y^2 = 8x$ by its latus rectum $x = 2$ if the density varies as the distance from the latus rectum. (See Fig. 74-4.)

Here, $\delta(x, y) = 2 - x$ and, by symmetry, $\bar{y} = 0$. For the upper half of the plate,

$$m = \iint_R \delta(x, y)\, dA = \int_0^4 \int_{y^2/8}^{2} k(2-x)\, dx\, dy = k\int_0^4 \left(2 - \frac{y^2}{4} + \frac{y^4}{128}\right) dy = \frac{64}{15}k$$

$$M_y = \iint_R \delta(x, y)x\, dA = \int_0^4 \int_{y^2/8}^{2} k(2-x)x\, dx\, dy = k\int_0^4 \left[\frac{4}{3} - \frac{y^4}{64} + \frac{y^6}{(24)(64)}\right] dy = \frac{128}{35}k$$

and $\bar{x} = M_y/m = \frac{6}{7}$. The center of mass has coördinates $(\frac{6}{7}, 0)$.

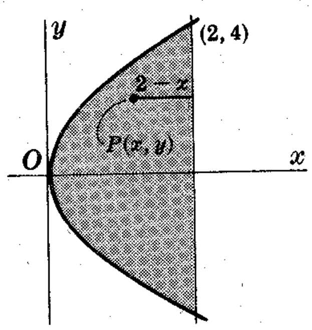

Fig. 74-4

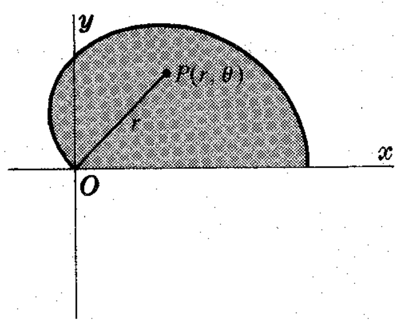

Fig. 74-5

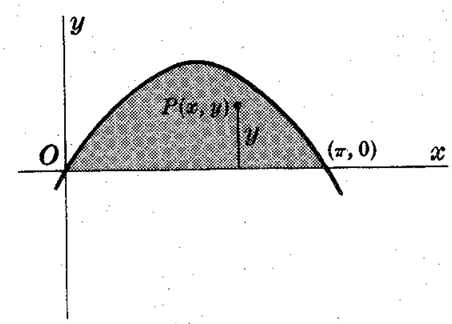

Fig. 74-6

5. Find the center of mass of a plate in the form of the upper half of the cardioid $r = 2(1 + \cos\theta)$ if the density varies as the distance from the pole. (See Fig. 74-5.)

$$m = \iint_R \delta(r, \theta)\, dA = \int_0^{\pi}\int_0^{2(1+\cos\theta)} (kr)r\, dr\, d\theta = \tfrac{8}{3}k\int_0^{\pi} (1+\cos\theta)^3\, d\theta = \tfrac{20}{3}k\pi$$

$$M_x = \iint_R \delta(r, \theta)y\, dA = \int_0^{\pi}\int_0^{2(1+\cos\theta)} (kr)(r\sin\theta)r\, dr\, d\theta$$

$$= 4k\int_0^{\pi} (1+\cos\theta)^4 \sin\theta\, d\theta = \tfrac{128}{5}k$$

$$M_y = \iint_R \delta(r, \theta)x\, dA = \int_0^{\pi}\int_0^{2(1+\cos\theta)} (kr)(r\cos\theta)r\, dr\, d\theta = 14k\pi$$

Then $\bar{x} = \dfrac{M_y}{m} = \dfrac{21}{10}$, $\bar{y} = \dfrac{M_x}{m} = \dfrac{96}{25\pi}$, and the center of mass has coordinates $\left(\dfrac{21}{10}, \dfrac{96}{25\pi}\right)$.

6. Find the moment of inertia with respect to the x axis of a plate having for edges one arch of the curve $y = \sin x$ and the x axis if its density varies as the distance from the x axis. (See Fig. 74-6.)

$$m = \iint\limits_R \delta(x, y)\, dA = \int_0^{\pi} \int_0^{\sin x} ky\, dy\, dx = \tfrac{1}{2}k \int_0^{\pi} \sin^2 x\, dx = \tfrac{1}{4}k\pi$$

$$I_x = \iint\limits_R \delta(x, y) y^2\, dA = \int_0^{\pi} \int_0^{\sin x} (ky)(y^2)\, dy\, dx = \tfrac{1}{4}k \int_0^{\pi} \sin^4 x\, dx = \tfrac{3}{32}k\pi = \tfrac{3}{8}m$$

7. Find the mass of a sphere of radius a if the density varies inversely as the square of the distance from the center.

Take the sphere as in Fig. 74-7. Then $\delta(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2} = \dfrac{k}{\rho^2}$ and

$$m = \iiint\limits_R \delta(x, y, z)\, dV = 8 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \frac{k}{\rho^2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 8ka \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi\, d\phi\, d\theta = 8ka \int_0^{\pi/2} d\theta = 4k\pi a \text{ units}$$

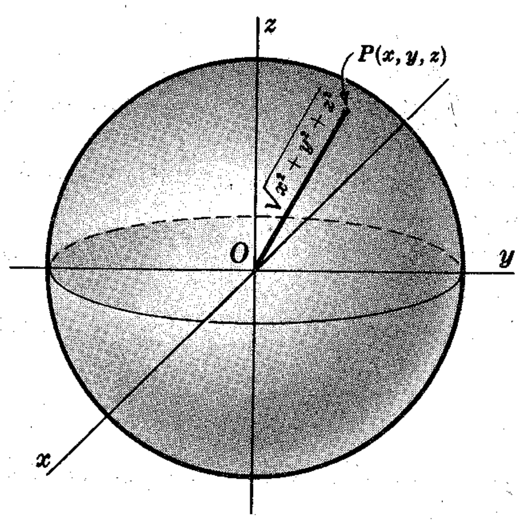

Fig. 74-7

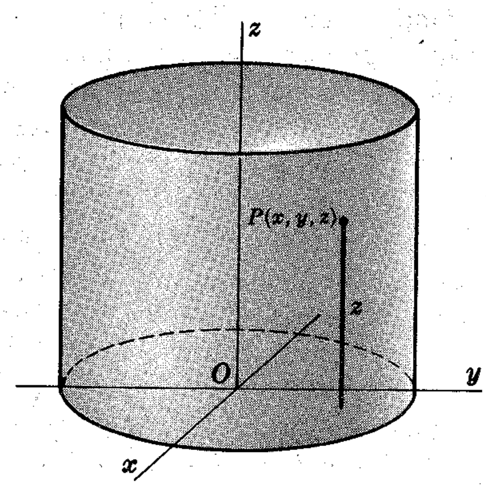

Fig. 74-8

8. Find the center of mass of a right circular cylinder of radius a and height h if the density varies as the distance from the base.

Take the cylinder as in Fig. 74-8, so that its equation is $r = a$ and the volume in question is that part of the cylinder between the planes $z = 0$ and $z = h$. Clearly, the center of mass lies on the z axis. Then

$$m = \iiint\limits_R \delta(z, r, \theta)\, dV = 4 \int_0^{\pi/2} \int_0^a \int_0^h (kz) r\, dz\, dr\, d\theta = 2kh^2 \int_0^{\pi/2} \int_0^a r\, dr\, d\theta$$

$$= kh^2a^2 \int_0^{\pi/2} d\theta = \tfrac{1}{2}k\pi h^2 a^2$$

$$M_{xy} = \iiint\limits_R \delta(z, r, \theta) z\, dV = 4 \int_0^{\pi/2} \int_0^a \int_0^h (kz^2) r\, dz\, dr\, d\theta = \tfrac{4}{3}kh^3 \int_0^{\pi/2} \int_0^a r\, dr\, d\theta$$

$$= \tfrac{2}{3}kh^3a^2 \int_0^{\pi/2} d\theta = \tfrac{1}{3}k\pi h^3 a^2$$

and $\bar{z} = M_{xy}/m = \tfrac{2}{3}h$. Thus the center of mass has coordinates $(0, 0, \tfrac{2}{3}h)$.

Supplementary Problems

9. Find the mass of:

(*a*) A straight rod of length a whose density varies as the square of the distance from one end *Ans.* $\frac{1}{3}ka^3$ units

(*b*) A plate in the form of a right triangle with legs a and b, if the density varies as the sum of the distances from the legs *Ans.* $\frac{1}{6}kab(a+b)$ units

(*c*) A circular plate of radius a whose density varies as the distance from the center *Ans.* $\frac{2}{3}ka^3\pi$ units

(*d*) A plate in the form of an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, if the density varies as the sum of the distances from its axes *Ans.* $\frac{4}{3}kab(a+b)$ units

(*e*) A circular cylinder of height b and radius of base a, if the density varies as the square of the distance from its axis *Ans.* $\frac{1}{2}ka^4b\pi$ units

(*f*) A sphere of radius a whose density varies as the distance from a fixed diametral plane *Ans.* $\frac{1}{2}ka^4\pi$ units

(*g*) A circular cone of height b and radius of base a whose density varies as the distance from its axis *Ans.* $\frac{1}{6}ka^3b\pi$ units

(*h*) A spherical surface whose density varies as the distance from a fixed diametral plane *Ans.* $2ka^3\pi$ units

10. Find the center of mass of:

(*a*) One quadrant of the plate of Problem 9(*c*) *Ans.* $(3a/2\pi, 3a/2\pi)$

(*b*) One quadrant of a circular plate of radius a, if the density varies as the distance from a bounding radius (the x axis) *Ans.* $(3a/8, 3a\pi/16)$

(*c*) A cube of edge a, if the density varies as the sum of the distances from three adjacent edges (on the coordinate axes) *Ans.* $(5a/9, 5a/9, 5a/9)$

(*d*) An octant of a sphere of radius a, if the density varies as the distance from one of the plane faces *Ans.* $(16a/15\pi, 16a/15\pi, 8a/15)$

(*e*) A right circular cone of height b and radius of base a, if the density varies as the distance from its base *Ans.* $(0, 0, 2b/5)$

11. Find the moment of inertia of:

(*a*) A square plate of side a with respect to a side, if the density varies as the square of the distance from an extremity of that side *Ans.* $\frac{7}{15}a^2m$

(*b*) A plate in the form of a circle of radius a with respect to its center, if the density varies as the square of the distance from the center *Ans.* $\frac{2}{3}a^2m$

(*c*) A cube of edge a with respect to an edge, if the density varies as the square of the distance from one extremity of that edge *Ans.* $\frac{38}{45}a^2m$

(*d*) A right circular cone of height b and radius of base a with respect to its axis, if the density varies as the distance from the axis *Ans.* $\frac{2}{5}a^2m$

(*e*) The cone of (*d*), if the density varies as the distance from the base *Ans.* $\frac{1}{5}a^2m$

Chapter 75

Differential Equations

A DIFFERENTIAL EQUATION is an equation that involves derivatives or differentials; examples are $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$ and $dy = (x + 2y)\,dx$.

The *order* of a differential equation is the order of the derivative of the highest order appearing in it. The first of the above equations is of order two, and the second is of order one. Both are said to be of *degree* one.

A *solution* of a differential equation is any relation between the variables that is free of derivatives or differentials and which satisfies the equation identically. The *general solution* of a differential equation of order n is that solution which contains the maximum number $(=n)$ of essential arbitrary constants. (See Problems 1 to 3.)

AN EQUATION OF THE FIRST ORDER AND DEGREE has the form $M(x, y)\,dx + N(x, y)\,dy = 0$. If such an equation has the particular form $f_1(x)g_2(y)\,dx + f_2(x)g_1(y)\,dy = 0$, the variables are *separable* and the solution is obtained as

$$\int \frac{f_1(x)}{f_2(x)}\,dx + \int \frac{g_1(y)}{g_2(y)}\,dy = C$$

(See Problems 4 to 6.)

A function $f(x, y)$ is said to be *homogeneous of degree n* in the variables if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. The equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be *homogeneous* if $M(x, y)$ and $N(x, y)$ are homogeneous of the same degree. The substitution

$$y = vx \qquad dy = v\,dx + x\,dv$$

will transform a homogeneous equation into one whose variables x and v are separable. (See Problems 7 to 9.)

CERTAIN DIFFERENTIAL EQUATIONS may be solved readily by taking advantage of the presence of integrable combinations of terms. An equation that is not immediately solvable by this method may be so solved after it is multiplied by a properly chosen function of x and y. This multiplier is called an *integrating factor* of the equation. (See Problems 10 to 14.)

The so-called *linear differential equation of the first order* $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone, has $\xi(x) = e^{\int P\,dx}$ as integrating factor. (See Problems 15 to 17.)

An equation of the form $\frac{dy}{dx} + Py = Qy^n$, where $n \neq 0, 1$, and where P and Q are functions of x alone, is reduced to the linear form by the substitution

$$y^{1-n} = z \qquad y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

(See Problems 18 to 19.)

Solved Problems

1. Show that (*a*) $y = 2e^x$, (*b*) $y = 3x$, and (*c*) $y = C_1e^x + C_2x$, where C_1 and C_2 are arbitrary constants, are solutions of the differential equation $y''(1-x) + y'x - y = 0$.

(*a*) Differentiate $y = 2e^x$ twice to obtain $y' = 2e^x$ and $y'' = 2e^x$. Substitute in the differential equation to obtain the identity $2e^x(1-x) + 2e^x x - 2e^x = 0$.
(*b*) Differentiate $y = 3x$ twice to obtain $y' = 3$ and $y'' = 0$. Substitute in the differential equation to obtain the identity $0(1-x) + 3x - 3x = 0$.
(*c*) Differentiate $y = C_1e^x + C_2x$ twice to obtain $y' = C_1e^x + C_2$ and $y'' = C_1e^x$. Substitute in the differential equation to obtain the identity $C_1e^x(1-x) + (C_1e^x + C_2)x - (C_1e^x + C_2x) = 0$.

Solution (*c*) is the *general solution* of the differential eqution because it satisfies the equation and contains the proper number of essential arbitrary constants. Solutions (*a*) and (*b*) are called *particular solutions* because each may be obtained by assigning particular values to the arbitrary constants of the general solution.

2. Form the differential equation whose general solution is (*a*) $y = Cx^2 - x$; (*b*) $y = C_1x^3 + C_2x + C_3$.

(*a*) Differentiate $y = Cx^2 - x$ once to obtain $y' = 2Cx - 1$. Solve for $C = \frac{1}{2}\left(\frac{y'+1}{x}\right)$ and substitute in the given relation (general solution) to obtain $y = \frac{1}{2}\left(\frac{y'+1}{x}\right)x^2 - x$ or $y'x = 2y + x$.
(*b*) Differentiate $y = C_1x^3 + C_2x + C_3$ three times to obtain $y' = 3C_1x^2 + C_2$, $y'' = 6C_1x$, $y''' = 6C_1$. Then $y'' = xy'''$ is the required equation. Note that the given relation is a solution of the equation $y^{(\mathrm{iv})} = 0$ but is not the general solution, since it contains only three arbitrary constants.

3. Form the second-order differential equation of all parabolas with principal axis along the x axis.

The system of parabolas has equation $y^2 = Ax + B$, where A and B are arbitrary constants. Differentiate twice to obtain $2yy' = A$ and $2yy'' + 2(y')^2 = 0$. The latter is the required equation.

4. Solve $\dfrac{dy}{dx} + \dfrac{1+y^3}{xy^2(1+x^2)} = 0$.

Here $xy^2(1+x^2)\,dy + (1+y^3)\,dx = 0$, or $\dfrac{y^2}{1+y^3}\,dy + \dfrac{1}{x(1+x^2)}\,dx = 0$ with the variables separated. Then partial-fraction decomposition yields

$$\frac{y^2\,dy}{1+y^3} + \frac{dx}{x} - \frac{x\,dx}{1+x^2} = 0,$$

and integration yields

$$\tfrac{1}{3}\ln|1+y^3| + \ln|x| - \tfrac{1}{2}\ln(1+x^2) = c$$

or

$$2\ln|1+y^3| + 6\ln|x| - 3\ln(1+x^2) = 6c$$

from which

$$\ln\frac{x^6(1+y^3)^2}{(1+x^2)^3} = 6c \quad\text{and}\quad \frac{x^6(1+y^3)^2}{(1+x^2)^3} = e^{6c} = C$$

5. Solve $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$.

Here $\dfrac{dy}{1+y^2} = \dfrac{dx}{1+x^2}$. Then integration yields arctan y = arctan x + arctan C, and

$$y = \tan(\arctan x + \arctan C) = \frac{x+C}{1-Cx}$$

6. Solve $\dfrac{dy}{dx} = \dfrac{\cos^2 y}{\sin^2 x}$.

The variables are easily separated to yield $\dfrac{dy}{\cos^2 y} = \dfrac{dx}{\sin^2 x}$ or $\sec^2 y\,dy = \csc^2 x\,dx$, and integration yields $\tan y = -\cot x + C$.

7. Solve $2xy\,dy = (x^2 - y^2)\,dx$.

The equation is homogeneous of degree two. The transformation $y = vx$, $dy = v\,dx + x\,dv$ yields $(2x)(vx)(v\,dx + x\,dv) = (x^2 - v^2x)\,dx$ or $\dfrac{2v\,dv}{1-3v^2} = \dfrac{dx}{x}$. Then integration yields

$$-\tfrac{1}{3}\ln|1-3v^2| = \ln|x| + \ln c$$

from which $\ln|1-3v^2| + 3\ln|x| + \ln C' = 0$ or $C'|x^3(1-3v^2)| = 1$.

Now $\pm C'x^3(1-3v^2) = Cx^3(1-3v^2) = 1$, and using $v = y/x$ produces $C(x^3 - 3xy^2) = 1$.

8. Solve $x\sin\dfrac{y}{x}\,(y\,dx + x\,dy) + y\cos\dfrac{y}{x}\,(x\,dy - y\,dx) = 0$.

The equation is homogeneous of degree two. The transformation $y = vx$, $dy = v\,dx + x\,dv$ yields

$$x\sin v(vx\,dx + x^2\,dv + vx\,dx) + vx\cos v(x^2\,dv + vx\,dx - vx\,dx) = 0$$

or

$$\sin v(2v\,dx + x\,dv) + xv\cos v\,dv = 0$$

or

$$\frac{\sin v + v\cos v}{v\sin v}\,dv + 2\,\frac{dx}{x} = 0$$

Then $\ln|v\sin v| + 2\ln|x| = \ln C'$, so that $x^2v\sin v = C$ and $xy\sin\dfrac{y}{x} = C$.

9. Solve $(x^2 - 2y^2)\,dy + 2xy\,dx = 0$.

The equation is homogeneous of degree two, and the standard transformation yields

$$(1-2v^2)(v\,dx + x\,dv) + 2v\,dx = 0$$

or

$$\frac{1-2v^2}{v(3-2v^2)}\,dv + \frac{dx}{x} = 0$$

or

$$\frac{dv}{3v} - \frac{4v\,dv}{3(3-2v^2)} + \frac{dx}{x} = 0$$

Integration yields $\tfrac{1}{3}\ln|v| + \tfrac{1}{3}\ln|3-2v^2| + \ln|x| = \ln c$, which we may write as $\ln|v| + \ln|3-2v^2| + 3\ln|x| = \ln C'$. Then $vx^3(3-2v^2) = C$ and $y(3x^2 - 2y^2) = C$.

10. Solve $(x^2 + y)\,dx + (y^3 + x)\,dy = 0$.

Integrate $x^2\,dx + (y\,dx + x\,dy) + y^3\,dy = 0$, term by term, to obtain $\dfrac{x^3}{3} + xy + \dfrac{y^4}{4} = C$.

11. Solve $(x + e^{-x}\sin y)\,dx - (y + e^{-x}\cos y)\,dy = 0$.

Integrate $x\,dx - y\,dy - (e^{-x}\cos y\,dy - e^{-x}\sin y\,dx) = 0$, term by term, to obtain

$$\tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 - e^{-x}\sin y = C$$

12. Solve $x\,dy - y\,dx = 2x^3\,dx$.

The combination $x\,dy - y\,dx$ suggests $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$. Hence, multiplying the given equation by $\xi(x) = \frac{1}{x^2}$, we obtain $\frac{x\,dy - y\,dx}{x^2} = 2x\,dx$, from which $\frac{y}{x} = x^2 + C$ or $y = x^3 + Cx$.

13. Solve $x\,dy + y\,dx = 2x^2y\,dx$.

The combination $x\,dy + y\,dx$ suggests $d(\ln xy) = \frac{x\,dy + y\,dx}{xy}$. Hence, multiplying the given equation by $\xi(x, y) = \frac{1}{xy}$, we obtain $\frac{x\,dy + y\,dx}{xy} = 2x\,dx$, from which $\ln|xy| = x^2 + C$.

14. Solve $x\,dy + (3y - e^x)\,dx = 0$.

Multiply the equation by $\xi(x) = x^2$ to obtain $x^3\,dy + 3x^2y\,dx = x^2e^x\,dx$. This yields

$$x^3y = \int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$$

15. Solve $\frac{dy}{dx} + \frac{2}{x}\,y = 6x^3$.

Here $P(x) = \frac{2}{x}$, $\int P(x)\,dx = \ln x^2$, and an integrating factor is $\xi(x) = e^{\ln x^2} = x^2$. We multiply the given equation by $\xi(x) = x^2$ to obtain $x^2\,dy + 2xy\,dx = 6x^5\,dx$. Then integration yields $x^2y = x^6 + C$.

Note 1: After multiplication by the integrating factor, the terms on the left side of the resulting equation are an *integrable combination*.

Note 2: The integrating factor for a given equation is not unique. In this problem x^2, $3x^2$, $\frac{1}{2}x^2$, etc., are all integrating factors. Hence, we write the simplest particular integral of $P(x)\,dx$ rather than the general integral, $\ln x^2 + \ln C = \ln Cx^2$.

16. Solve $\tan x\,\frac{dy}{dx} + y = \sec x$.

Since $\frac{dy}{dx} + y\cot x = \csc x$, we have $\int P(x)\,dx = \int \cot x\,dx = \ln|\sin x|$, and $\xi(x) = e^{\ln|\sin x|} = |\sin x|$. Then multiplication by $\xi(x)$ yields

$$\sin x\left(\frac{dy}{dx} + y\cot x\right) = \sin x\csc x \qquad \text{or} \qquad \sin x\,dy + y\cos x\,dx = dx$$

and integration gives $y\sin x = x + C$.

17. Solve $\frac{dy}{dx} - xy = x$.

Here $P(x) = -x$, $\int P(x)\,dx = -\frac{1}{2}x^2$, and $\xi(x) = e^{-\frac{1}{2}x^2}$. This produces

$$e^{-\frac{1}{2}x^2}\,dy - xye^{-\frac{1}{2}x^2}\,dx = xe^{-\frac{1}{2}x^2}\,dx$$

and integration yields $ye^{-\frac{1}{2}x^2} = -e^{-\frac{1}{2}x^2} + C$, or $y = Ce^{\frac{1}{2}x^2} - 1$.

18. Solve $\frac{dy}{dx} + y = xy^2$.

The equation is of the form $\frac{dy}{dx} + Py = Qy^n$, with $n = 2$. Hence we use the substitution $y^{1-n} = y^{-1} = z$, $y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$. For convenience, we write the original equation in the form $y^{-2}\frac{dy}{dx} + y^{-1} = x$, obtaining $-\frac{dz}{dx} + z = x$ or $\frac{dz}{dx} - z = -x$.

The integrating factor is $\xi(x) = e^{\int P\,dx} = e^{-\int dx} = e^{-x}$. It gives us $e^{-x}\,dz - ze^{-x}\,dx = -xe^{-x}\,dx$, from which $ze^{-x} = xe^{-x} + e^{-x} + C$. Finally, since $z = y^{-1}$, we have $\frac{1}{y} = x + 1 + Ce^x$.

19. Solve $\frac{dy}{dx} + y\tan x = y^3 \sec x$.

Write the equation in the form $y^{-3}\frac{dy}{dx} + y^{-2}\tan x = \sec x$. Then use the substitution $y^{-2} = z$, $y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}$ to obtain $\frac{dz}{dx} - 2z\tan x = -2\sec x$.

The integrating factor is $\xi(x) = e^{-2\int \tan x\,dx} = \cos^2 x$. It gives $\cos^2 x\,dz - 2z\cos x \sin x\,dx = -2\cos x\,dx$, from which $z\cos^2 x = -2\sin x + C$, or $\frac{\cos^2 x}{y^2} = -2\sin x + C$.

20. When a bullet is fired into a sand bank, its retardation is assumed equal to the square root of its velocity on entering. For how long will it travel if its velocity on entering the bank is 144 ft/sec?

Let v represent the bullet's velocity t seconds after striking the bank. Then the retardation is $-\frac{dv}{dt} = \sqrt{v}$, so $\frac{dv}{\sqrt{v}} = -dt$ and $2\sqrt{v} = -t + C$.

When $t = 0$, $v = 144$ and $C = 2\sqrt{144} = 24$. Thus, $2\sqrt{v} = -t + 24$ is the law governing the motion of the bullet. When $v = 0$, $t = 24$; the bullet will travel 24 seconds before coming to rest.

21. A tank contains 100 gal of brine holding 200 lb of salt in solution. Water containing 1 lb of salt per gallon flows into the tank at the rate of 3 gal/min, and the mixture, kept uniform by stirring, flows out at the same rate. Find the amount of salt at the end of 90 min.

Let q denote the number of pounds of salt in the tank at the end of t minutes. Then $\frac{dq}{dt}$ is the rate of change of the amount of salt at time t.

Three pounds of salt enters the tank each minute, and $0.03q$ pounds is removed. Thus, $\frac{dq}{dt} = 3 - 0.03q$. Rearranged, this becomes $\frac{dq}{3 - 0.03q} = dt$, and integration yields $\frac{\ln(0.03q - 3)}{0.03} = -t + C$.

When $t = 0$, $q = 200$ and $C = \frac{\ln 3}{0.03}$ so that $\ln(0.03q - 3) = -0.03t + \ln 3$. Then $0.01q - 1 = e^{-0.03t}$, and $q = 100 + 100e^{-0.03t}$. When $t = 90$, $q = 100 + 100e^{-2.7} = 106.72$ lb.

22. Under certain conditions, cane sugar in water is converted into dextrose at a rate proportional to the amount that is unconverted at any time. If, of 75 grams at time $t = 0$, 8 grams are converted during the first 30 min, find the amount converted in $1\frac{1}{2}$ hours.

Let q denote the amount converted in t minutes. Then $\frac{dq}{dt} = k(75 - q)$, from which $\frac{dq}{75 - q} = k\,dt$, and integration gives $\ln(75 - q) = -kt + C$.

When $t = 0$, $q = 0$ and $C = \ln 75$, so that $\ln(75 - q) = -kt + \ln 75$.

When $t = 30$ and $q = 8$, we have $30k = \ln 75 - \ln 67$; hence, $k = 0.0038$, and $q = 75(1 - e^{-0.0038t})$.

When $t = 90$, $q = 75(1 - e^{-0.34}) = 21.6$ grams.

Supplementary Problems

23. Form the differential equation whose general solution is:

(*a*) $y = Cx^2 + 1$ (*b*) $y = C^2x + C$ (*c*) $y = Cx^2 + C^2$
(*d*) $xy = x^3 - C$ (*e*) $y = C_1 + C_2x + C_3x^2$ (*f*) $y = C_1e^x + C_2e^{2x}$
(*g*) $y = C_1 \sin x + C_2 \cos x$ (*h*) $y = C_1e^x \cos(3x + C_2)$

Ans. (*a*) $xy' = 2(y - 1)$; (*b*) $y' = (y - xy')^2$; (*c*) $4x^2y = 2x^3y' + (y')^2$; (*d*) $xy' + y = 3x^2$; (*e*) $y''' = 0$; (*f*) $y'' - 3y' + 2y = 0$; (*g*) $y'' + y = 0$; (*h*) $y'' - 2y' + 10y = 0$

24. Solve:

(*a*) $y\,dy - 4x\,dx = 0$ *Ans.* $y^2 = 4x^2 + C$
(*b*) $y^2\,dy - 3x^5\,dx = 0$ *Ans.* $2y^3 = 3x^6 + C$
(*c*) $x^3y' = y^2(x - 4)$ *Ans.* $x^2 - xy + 2y = Cx^2y$
(*d*) $(x - 2y)\,dy + (y + 4x)\,dx = 0$ *Ans.* $xy - y^2 + 2x^2 = C$
(*e*) $(2y^2 + 1)y' = 3x^2y$ *Ans.* $y^2 + \ln|y| = x^3 + C$
(*f*) $xy'(2y - 1) = y(1 - x)$ *Ans.* $\ln|xy| = x + 2y + C$
(*g*) $(x^2 + y^2)\,dx = 2xy\,dy$ *Ans.* $x^2 - y^2 = Cx$
(*h*) $(x + y)\,dy = (x - y)\,dx$ *Ans.* $x^2 - 2xy - y^2 = C$
(*i*) $x(x + y)\,dy - y^2\,dx = 0$ *Ans.* $y = Ce^{-y/x}$
(*j*) $x\,dy - y\,dx + xe^{-y/x}\,dx = 0$ *Ans.* $e^{y/x} + \ln|Cx| = 0$
(*k*) $dy = (3y + e^{2x})\,dx$ *Ans.* $y = (Ce^x - 1)e^{2x}$
(*l*) $x^2y^2\,dy = (1 - xy^3)\,dx$ *Ans.* $2x^3y^3 = 3x^2 + C$

25. The tangent and normal to a curve at point $P(x, y)$ meet the x axis in T and N, respectively, and the y axis in S and M, respectively. Determine the family of curves satisfying the condition:

(*a*) $TP = PS$ (*b*) $NM = MP$ (*c*) $TP = OP$ (*d*) $NP = OP$

Ans. (*a*) $xy = C$; (*b*) $2x^2 + y^2 = C$; (*c*) $xy = C$, $y = Cx$; (*d*) $x^2 \pm y^2 = C$

26. Solve Problem 21, assuming that pure water flows into the tank at the rate 3 gal/min and the mixture flows out at the same rate. *Ans.* 13.44 lb

27. Solve Problem 26 assuming that the mixture flows out at the rate 4 gal/min. $\left(\textit{Hint}: dq = -\dfrac{4q}{100 - t}\,dt\right)$ *Ans.* 0.02 lb

Chapter 76

Differential Equations of Order Two

THE SECOND-ORDER DIFFERENTIAL EQUATIONS that we shall solve in this chapter are of the following types:

$\dfrac{d^2y}{dx^2} = f(x)$ (See Problem 1.)

$\dfrac{d^2y}{dx^2} = f\left(x, \dfrac{dy}{dx}\right)$ (See Problems 2 and 3.)

$\dfrac{d^2y}{dx^2} = f(y)$ (See Problems 4 and 5.)

$\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, where P and Q are constants and R is a constant or function of x only (See Problems 6 to 11.)

If the equation $m^2 + Pm + Q = 0$ has two *distinct* roots m_1 and m_2, then $y = C_1e^{m_1x} + C_2e^{m_2x}$ is the general solution of the equation $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = 0$. If the two roots are identical so that $m_1 = m_2 = m$, then $y = C_1e^{mx} + C_2xe^{mx}$ is the general solution.

The general solution of $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = 0$ is called the *complementary function* of the equation $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R(x)$. If $f(x)$ satisfies the latter equation, then $y =$ complementary function $+ f(x)$ is its general solution. The function $f(x)$ is called a *particular solution*.

Solved Problems

1. Solve $\dfrac{d^2y}{dx^2} = xe^x + \cos x$.

Here $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = xe^x + \cos x$. Hence, $\dfrac{dy}{dx} = \int (xe^x + \cos x)\,dx = xe^x - e^x + \sin x + C_1$, and another integration yields $y = xe^x - 2e^x - \cos x + C_1x + C_2$.

2. Solve $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = a$.

Let $p = \dfrac{dy}{dx}$; then $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$ and the given equation becomes $x^2\dfrac{dp}{dx} + xp = a$ or $x\,dp + p\,dx = \dfrac{a}{x}\,dx$. Then integration yields $xp = a\ln|x| + C_1$, or $x\dfrac{dy}{dx} = a\ln|x| + C_1$. When this is written as $dy = a\ln|x|\dfrac{dx}{x} + C_1\dfrac{dx}{x}$, integration gives $y = \frac{1}{2}a\ln^2|x| + C_1\ln|x| + C_2$.

3. Solve $xy'' + y' + x = 0$.

Let $p = \frac{dy}{dx}$. Then $\frac{d^2y}{dx^2} = \frac{dp}{dx}$ and the given equation becomes $x\,\frac{dp}{dx} + p + x = 0$ or $x\,dp + p\,dx = -x\,dx$. Integration gives $xp = -\frac{1}{2}x^2 + C_1$, substitution for p gives $\frac{dy}{dx} = -\frac{1}{2}x + \frac{C_1}{x}$, and another integration yields $y = -\frac{1}{4}x^2 + C_1 \ln|x| + C_2$.

4. Solve $\frac{d^2y}{dx^2} - 2y = 0$.

Since $\frac{d}{dx}[(y')^2] = 2y'y''$, we can multiply the given equation by $2y'$ to obtain $2y'y'' = 4yy'$, and integrate to obtain $(y')^2 = 4\int yy'\,dx = 4\int y\,dy = 2y^2 + C_1$.

Then $\frac{dy}{dx} = \sqrt{2y^2 + C_1}$, so that $\frac{dy}{\sqrt{2y^2 + C_1}} = dx$ and $\ln|\sqrt{2}y + \sqrt{2y^2 + C_1}| = \sqrt{2}x + \ln C_2'$. The last equation yields $\sqrt{2}y + \sqrt{2y^2 + C_1} = C_2 e^{\sqrt{2}x}$.

5. Solve $y'' = -1/y^3$.

Multiply by $2y'$ to obtain $2y'y'' = -\frac{2y'}{y^3}$. Then integration yields

$$(y')^2 + \frac{1}{y^2} + C_1 \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{\sqrt{1 + C_1y^2}}{y} \qquad \text{or} \qquad \frac{y\,dy}{\sqrt{1 + C_1y^2}} = dx$$

Another integration gives $\sqrt{1 + C_1y^2} = C_1x + C_2$, or $(C_1x + C_2)^2 - C_1y^2 = 1$.

6. Solve $\frac{d^2y}{dx^2} + 3\,\frac{dy}{dx} - 4y = 0$.

Here we have $m^2 + 3m - 4 = 0$, from which $m = 1, -4$. The general solution is $y = C_1e^x + C_2e^{-4x}$.

7. Solve $\frac{d^2y}{dx^2} + 3\,\frac{dy}{dx} = 0$.

Here $m^2 + 3m = 0$, from which $m = 0, -3$. The general solution is $y = C_1 + C_2e^{-3x}$.

8. Solve $\frac{d^2y}{dx^2} - 4\,\frac{dy}{dx} + 13y = 0$.

Here $m^2 - 4m + 13 = 0$, with roots $m_1 = 2 + 3i$ and $m_2 = 2 - 3i$. The general solution is

$$y = C_1e^{(2+3i)x} + C_2e^{(2-3i)x} = e^{2x}(C_1e^{3ix} + C_2e^{-3ix})$$

Since $e^{iax} = \cos ax + i \sin ax$, we have $e^{3ix} = \cos 3x + i \sin 3x$ and $e^{-3ix} = \cos 3x - i \sin 3x$. Hence, the solution may be put in the form

$$\begin{aligned} y &= e^{2x}[C_1(\cos 3x + i \sin 3x) + C_2(\cos 3x - i \sin 3x)] \\ &= e^{2x}[(C_1 + C_2)\cos 3x + i(C_1 - C_2)\sin 3x] \\ &= e^{2x}(A \cos 3x + B \sin 3x) \end{aligned}$$

9. Solve $\frac{d^2y}{dx^2} - 4\,\frac{dy}{dx} + 4y = 0$.

Here $m^2 - 4m + 4 = 0$, with roots $m = 2, 2$. The general solution is $y = C_1e^{2x} + C_2xe^{2x}$.

10. Solve $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 4y = x^2$.

From Problem 6, the complementary function is $y = C_1e^x + C_2e^{-4x}$.

To find a particular solution of the equation, we note that the right-hand member is $R(x) = x^2$. This suggests that the particular solution will contain a term in x^2 and perhaps other terms obtained by successive differentiation. We assume it to be of the form $y = Ax^2 + Bx + C$, where the constants A, B, C are to be determined. Hence we substitute $y = Ax^2 + Bx + C$, $y' = 2Ax + B$, and $y'' = 2A$ in the differential equation to obtain

$$2A + 3(2Ax + B) - 4(Ax^2 + Bx + C) = x^2 \quad \text{or} \quad -4Ax^2 + (6A - 4B)x + (2A + 3B - 4C) = x^2$$

Since this latter equation is an identity in x, we have $-4A = 1$, $6A - 4B = 0$, and $2A + 3B - 4C = 0$. These yield $A = -\frac{1}{4}$, $B = -\frac{3}{8}$, $C = -\frac{13}{32}$, and $y = -\frac{1}{4}x^2 - \frac{3}{8}x - \frac{13}{32}$ is a particular solution. Thus, the general solution is $y = C_1e^x + C_2e^{-4x} - \frac{1}{4}x^2 - \frac{3}{8}x - \frac{13}{32}$.

11. Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = \cos x$.

Here $m^2 - 2m - 3 = 0$, from which $m = -1, 3$; the complementary function is $y = C_1e^{-x} + C_2e^{3x}$. The right-hand member of the differential equation suggests that a particular solution is of the form $A\cos x + B\sin x$. Hence, we substitute $y = A\cos x + B\sin x$, $y' = B\cos x - A\sin x$, and $y'' = -A\cos x - B\sin x$ in the differential equation to obtain

$$(-A\cos x - B\sin x) - 2(B\cos x - A\sin x) - 3(A\cos x + B\sin x) = \cos x$$

or

$$-2(2A + B)\cos x + 2(A - 2B)\sin x = \cos x$$

The latter equation yields $-2(2A + B) = 1$ and $A - 2B = 0$, from which $A = -\frac{1}{5}$, $B = -\frac{1}{10}$. The general solution is $C_1e^{-x} + C_2e^{3x} - \frac{1}{5}\cos x - \frac{1}{10}\sin x$.

12. A weight attached to a spring moves up and down, so that the equation of motion is $\dfrac{d^2s}{dt^2} + 16s = 0$, where s is the stretch of the spring at time t. If $s = 2$ and $\dfrac{ds}{dt} = 1$ when $t = 0$, find s in terms of t.

Here $m^2 + 16 = 0$ yields $m = \pm 4i$, and the general solution is $s = A\cos 4t + B\sin 4t$. Now when $t = 0$, $s = 2 = A$, so that $s = 2\cos 4t + B\sin 4t$.

Also when $t = 0$, $ds/dt = 1 = -8\sin 4t + 4B\cos 4t = 4B$, so that $B = \frac{1}{4}$. Thus, the required equation is $s = 2\cos 4t + \frac{1}{4}\sin 4t$.

13. The electric current in a certain circuit is given by $\dfrac{d^2I}{dt^2} + 4\dfrac{dI}{dt} + 2504I = 110$. If $I = 0$ and $\dfrac{dI}{dt} = 0$ when $t = 0$, find I in terms of t.

Here $m^2 + 4m + 2504 = 0$ yields $m = -2 + 50i$, $-2 - 50i$; the complementary function is $e^{-2t}(A\cos 50t + B\sin 50t)$. Because the right-hand member is a constant, we find that the particular solution is $I = 110/2504 = 0.044$. Thus, the general solution is $I = e^{-2t}(A\cos 50t + B\sin 50t) + 0.044$.

When $t = 0$, $I = 0 = A + 0.044$; then $A = -0.044$.

Also when $t = 0$, $dI/dt = 0 = e^{-2t}[(-2A + 50B)\cos 50t - (2B + 50A)\sin 50t] = -2A + 50B$. Then $B = -0.0018$, and the required relation is $I = -e^{-2t}(0.044\cos 50t + 0.0018\sin 50t) + 0.044$.

14. A chain 4 ft long starts to slide off a flat roof with 1 ft hanging over the edge. Discounting friction, find (*a*) the velocity with which it slides off and (*b*) the time required to slide off.

Let s denote the length of the chain hanging over the edge of the roof at time t.

(*a*) The force F causing the chain to slide off the roof is the weight of the part hanging over the edge.

That weight is $mgs/4$. Hence,

$$F = \text{mass} \times \text{acceleration} = ms'' = \tfrac{1}{4}mgs \qquad \text{or} \qquad s'' = \tfrac{1}{4}gs$$

Multiplying by $2s'$ yields $2s's'' = \frac{1}{2}gss'$ and integrating once gives $(s')^2 = \frac{1}{4}gs^2 + C_1$.

When $t = 0$, $s = 1$ and $s' = 0$. Hence, $C_1 = -\frac{1}{4}g$ and $s' = \frac{1}{2}\sqrt{g}\sqrt{s^2 - 1}$. When $s = 4$, $s' = \frac{1}{2}\sqrt{15g}$ ft/sec.

(*b*) Since $\dfrac{ds}{\sqrt{s^2 - 1}} = \frac{1}{2}\sqrt{g}\,dt$, integration yields $\ln|s + \sqrt{s^2 - 1}| = \frac{1}{2}\sqrt{g}t + C_2$. When $t = 0$, $s = 1$. Then $C_2 = 0$ and $\ln(s + \sqrt{s^2 - 1}) = \frac{1}{2}\sqrt{g}t$.

When $s = 4$, $t = \dfrac{2}{\sqrt{g}} \ln(4 + \sqrt{15})$ sec.

15. A boat of mass 1600 lb has a speed of 20 ft/sec when its engine is suddenly stopped (at $t = 0$). The resistance of the water is proportional to the speed of the boat and is 200 lb when $t = 0$. How far will the boat have moved when its speed is reduced to 5 ft/sec?

Let s denote the distance traveled by the boat t seconds after the engine is stopped. Then the force F on the boat is

$$F = ms'' = -Ks' \qquad \text{from which} \qquad s'' = -ks'$$

To determine k, we note that at $t = 0$, $s' = 20$ and $s'' = \dfrac{\text{force}}{\text{mass}} = -\dfrac{200g}{1600} = -4$. Then $k = -s''/s' = \frac{1}{5}$. Now $s'' = \dfrac{dv}{dt} = -\dfrac{v}{5}$, and integration gives $\ln v = -\frac{1}{5}t + C_1$, or $v = C_1 e^{-t/5}$.

When $t = 0$, $v = 20$. Then $C_1 = 20$ and $v = \dfrac{ds}{dt} = 20e^{-t/5}$. Another integration yields $s = -100e^{-t/5} + C_2$.

When $t = 0$, $s = 0$; then $C_2 = 100$ and $s = 100(1 - e^{-t/5})$. We require the value of s when $v = 5 = 20e^{-t/5}$, that is, when $e^{-t/5} = \frac{1}{4}$. Then $s = 100(1 - \frac{1}{4}) = 75$ ft.

Supplementary Problems

In Problems 16 to 32, solve the given equation.

16. $\dfrac{d^2y}{dx^2} = 3x + 2$ *Ans.* $y = \frac{1}{2}x^3 + x^2 + C_1x + C_2$

17. $e^{2x}\dfrac{d^2y}{dx^2} = 4(e^{4x} + 1)$ *Ans.* $y = e^{2x} + e^{-2x} + C_1x + C_2$

18. $\dfrac{d^2y}{dx^2} = -9\sin 3x$ *Ans.* $y = \sin 3x + C_1x + C_2$

19. $x\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 4x = 0$ *Ans.* $y = x^2 + C_1x^4 + C_2$

20. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 2x - x^2$ *Ans.* $y = \dfrac{x^3}{3} + C_1e^x + C_2$

21. $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 8x^3$ *Ans.* $y = x^4 + C_1x^2 + C_2$

22. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$ *Ans.* $y = C_1e^x + C_2e^{2x}$

23. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$ *Ans.* $y = C_1e^{-2x} + C_2e^{-3x}$

24. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0$ *Ans.* $y = C_1 + C_2e^x$

25. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$ *Ans.* $y = C_1xe^x + C_2e^x$

26. $\dfrac{d^2y}{dx^2} + 9y = 0$ *Ans.* $y = C_1 \cos 3x + C_2 \sin 3x$

27. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = 0$ *Ans.* $y = e^x(C_1 \cos 2x + C_2 \sin 2x)$

28. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = 0$ *Ans.* $y = e^{2x}(C_1 \cos x + C_2 \sin x)$

29. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 3y = 6x + 23$ *Ans.* $y = C_1e^{-x} + C_2e^{-3x} + 2x + 5$

30. $\dfrac{d^2y}{dx^2} + 4y = e^{3x}$ *Ans.* $y = C_1 \sin 2x + C_2 \cos 2x + \dfrac{e^{3x}}{13}$

31. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = x + e^{2x}$ *Ans.* $y = C_1e^{3x} + C_2xe^{3x} + e^{2x} + \dfrac{x}{9} + \dfrac{2}{27}$

32. $\dfrac{d^2y}{dx^2} - y = \cos 2x - 2\sin 2x$ *Ans.* $y = C_1e^x + C_2e^{-x} - \frac{1}{5}\cos 2x + \frac{2}{5}\sin 2x$

33. A particle of mass m, moving in a medium that offers a resistance proportional to the velocity, is subject to an attracting force proportional to the displacement. Find the equation of motion of the particle if at time $t = 0$, $s = 0$ and $s' = v_0$. $\left(\textit{Hint}: \text{Here } m\dfrac{d^2s}{dt^2} = -k_1\dfrac{ds}{dt} - k_2s \text{ or } \dfrac{d^2s}{dt^2} + 2b\dfrac{ds}{dt} + c^2s = 0,\ b > 0.\right)$

Ans. If $b^2 = c^2$, $s = v_0te^{-bt}$; if $b^2 < c^2$, $s = \dfrac{v_0}{\sqrt{c^2 - b^2}}e^{-bt}\sin\sqrt{c^2 - b^2}t$; if $b^2 > c^2$,
$s = \dfrac{v_0}{2\sqrt{b^2 - c^2}} \cdot \left(e^{(-b+\sqrt{b^2-c^2})t} - e^{(-b-\sqrt{b^2-c^2})t}\right)$

Index

Abscissa, 8
Absolute convergence, 345
Absolute value, 1
Absolute maximum and miminum, 106
Acceleration:
  angular, 112
  in curvilinear motion, 165
  in rectilinear motion, 112, 247
  tangential and normal components of, 166
  vector, 165
Alternating series, 345
Analytic proofs of geometric theorems, 10
Angle:
  between two curves, 91, 173
  of inclination, 172
Angular velocity and acceleration, 112
Antiderivative, 206
Approximate integration, 375
Approximations, 371
  by differentials, 196
  by the law of the mean, 185
  of roots, 197
Arc length, 148, 321
  by integration, 305
  derivative of, 148, 174
Area:
  by integration, 260
  in polar coordinates, 316
  of a curved surface, 451
  of a surface of revolution, 309
Asymptote, 40, 201
Average rate of change, 73
Average ordinate, 267
Axis of rotation, 272

Binormal vector, 424
Bliss's theorem, 252

Center of curvature, 149
Centroid:
  of arcs, 213, 321
  of plane area, 284, 316, 442
  of surface of revolution, 321
  of volume, 457
Chain rule, 80, 386
Circle, 31
  equation of, 31
  of curvature, 149
  osculating, 149
Circular motion, 112
Comparison test, 338
Complementary function, 476
Completing the square, 32
Components:
  of acceleration, 166
  of a vector, 156
Composite functions, 80
Concavity, 97, 187
Conditional convergence, 345
Conic sections, 40
Constant of integration, 206
Convergence of series, 333
  absolute, 345
  conditional, 345
Coordinate system:
  cylindrical and spherical, 456
  linear, 1
  rectangular, 8
  polar, 172
Continuity, 68, 380
Cosecant, 120
Cosine, 120
Cotangent, 120
Critical numbers, points, values, 96
Cubic curve, 42
Curl, 427
Curvature, 148, 174
Curve tracing, 201
Curvilinear motion, 165, 174
Cylindrical coordinates, 456

Decay constant, 269
Definite integral, 251
Delta neighborhood, 4
Derivative, 73
  directional, 417
  higher order, 81, 88
  of arc length, 148, 174
  partial, 380
  second, 81
  total, 387
Differentiability, 74
Differential, 196
  approximation by, 196
  partial, 386
  total, 386
Differential equations, 470
  second-order, 476

Differentiation, 79
  implicit, 88
  logarithmic, 133
  of exponential functions, 133, 269
  of hyperbolic functions, 141
  of inverse hyperbolic functions, 141
  of inverse trigonometric functions, 129
  of logarithmic functions, 133, 268
  of trigonometric functions, 122
  of vector functions, 158
  partial, 380
Direction cosines, 399, 402
Direction numbers, 402
Directional derivative, 417
Directrix of a parabola, 43
Disc method, 272
Discontinuity, 68
Distance formula, 10
Divergence of a vector function, 427
Domain of a function, 52
Double integral, 435

e, 133, 268
Ellipses, 39
  center, eccentricity, foci, major axis, minor axis of, 45
Equations, graphs of, 39
Evolute, 149
Expansion in power series, 360
Exponential functions, 133, 269
Exponential growth and decay, 269
Extended law of the mean, 184
Extent of a graph, 201

First-derivative test, 97
Fluid pressure, 297
Focus of a parabola, 43
Foci:
  of an ellipse, 45
  of a hyperbola, 46
Force, work done by a, 301
Functions, 52
  composite, 80
  continuous, 68
  decreasing, 96
  domain of, 52
  exponential, 133, 269
  graphs of, 53
  homogeneous, 393
  hyperbolic, 141
  implicit, 88, 394
  increasing, 96
  inverse, 79
  inverse trigonometric, 129

Functions (*continued*)
  logarithmic, 133, 268
  one-to-one, 79
  range of, 52
  trigonometric, 120
Fundamental theorem of integral calculus, 252

Generalized law of the mean, 184
Gradient, 417, 426
Graphs of equations, 39
Graphs of functions, 53
Growth constant, 269

Halflife, 269
Homogeneous bodies, 284
Homogeneous:
  equation, 470
  function, 393, 470
Hyperbolas, 40
  asymptotes of, 40
  center, conjugate axes, eccentricity, foci, transverse axes, vertices of, 46
  equilateral, 49
Hyperbolic functions, 141
  differentiation, 141
  integration of, 244
  inverse, 141

Implicit differentiation, 88
Implicit functions, 88, 394
Improper integrals, 326
Increment, 73
Indefinite integral, 206
Indeterminate forms, 190
Inequalities, 2
Infinite sequences, 58, 322
  general term of, 58
  limit of, 58
Infinite series, 333
Infinity, 60
Inflection point, 97
Instantaneous rate of change, 73
Integrand, 206, 251
Integral:
  definite, 251
  double, 435
  improper, 326
  indefinite, 206
  iterated, 436
  line, 427
  test for convergence, 338
  triple, 456
Integrating factor, 470

Integration:
approximate, 375
by partial fractions, 234
by parts, 219
by substitution, 207, 239
by trigonometric substitutions, 230
of exponential functions, 269
of hyperbolic functions, 244
of trigonometric functions, 225
standard formulas of, 206
Intercepts, 201
Interval of convergence, 354
Intervals, 2
Inverse function, 79
Inverse hyperbolic functions, 141
Inverse trigonometric functions, 129
Irreducible polynomial, 234
Iterated integral, 436

Latus rectum of a parabola, 43
Law of the mean, 183
Length of arc, 305, 321
L'Hospital's rule, 190
Limit:
of a function, 58, 380
of a sequence, 58
right and left, 59
Line, 17, 402
equations of, 19, 402
slope of, 17
Line integral, 427
Linear differential equation of the first order, 470
Logarithmic differentiation, 133
Logarithmic functions, 133, 268

Maclaurin's formula, 367
Maclaurin's series, 360
Mass, 284, 466
Maximum and minimum:
applications, 106
of functions of a single variable, 96
of functions of several variables, 418
Mean, law of the, 183
Midpoint formulas, 10
Moments of plane areas and solids, 284
Moments of inertia:
of arcs, 213
of plane area, 292, 442
of surface of revolution, 213
of volume, 292, 458
Motion:
circular, 112
curvilinear, 165
rectilinear, 112

Natural logarithm, 268
Normal line to a plane curve, 91
equation of, 91
Normal line to a surface, 411
Normal plane to a space curve, 411, 424

Octants, 398
One-to-one function, 79
Operations on series, 349
Ordinate, 8
Osculating circle, 149
Osculating plane, 424

Pappus, theorems of, 285, 213
Parabolas, 39
focus, directrix, latus rectum, vertex of, 43
Parallel-axis theorem, 292
Parametric equations, 145, 424
Partial derivatives, 380
Partial differential, 386
Partial fraction, 234
Particular solution, 476
Plane, 402
Point of inflection, 97
Point-slope equation of a line, 20
Polar coordinates, 172
Pole, 172
Polynomial test for convergence, 343
Position vector, 398
Power series, 354
approximations using, 372, 376
differentiation of, 355
integration of, 355
interval of convergence of, 354
uniform convergence of, 355
Principal normal, 424
Prismoidal formula, 376

Quadrants, 9

Radian measure, 120
Radius:
of curvature, 148
of gyration, 292
Range of a function, 52
Rate of change, 73
Ratio test, 338, 345
Real numbers, 1
Rectangular coordinate system, 8
Rectifying plane, 424
Rectilinear motion, 112
Reduction formulas, 219

Related rates, 116
Relative maximum and minimum, 96, 418
Remainder after n terms of a series, 350, 367
Right-handed system, 398
Rolle's theorem, 183
Root test for convergence, 344

Scalars, 155
Secant function, 120
Second-derivative test, 97
Sequences, 58, 332
  bounded, 332
  convergent and divergent, 332
  limit of, 332
  nondecreasing and nonincreasing, 332
Series, infinite, 333
  alternating, 345
  computations with, 371
  convergent, 333
  divergent, 333
  geometric, 335
  harmonic, 335
  Maclaurin's, 360
  partial sums of, 333, 354
  positive, 338
  power, 354
  remainder after n terms of, 354
  Taylor's, 360
  sum of, 333
  terms of, 333
Shell method, 274
Simpson's rule, 376
Sine, 120
Slope of a line, 17
Slope-intercept equation of a line, 20
Solid of revolution, 272
Space curve, 411, 423
Space vectors, 398
Speed, 112, 165
Spherical coordinates, 456
Stationary points, 96
Surface, 411, 424
Surface of revolution, 309
Symmetry, 201

Tangent function, 120
Tangent line to a plane curve, 91
  equation of, 91
Tangent line to a space curve, 411
Tangent plane to a surface, 411
Taylor's formula, 367
Taylor's series, 360
Time rate of change, 116
Total derivative, 387
Total differential, 386
Trapezoidal rule, 375
Triangle inequality, 1
Trigonometric functions, 120
  differentiation of, 122
Trigonometric integrals, 225
Trigonometric substitutions, 230
Triple integral, 456

Uniform convergence, 355

Variables, separable, 470
Vector(s):
  acceleration, 165
  addition of, 155
  components of, 156
  cross product of, 400
  direction cosines of, 399
  dot product of, 157
  equation of a line, 402
  equation of a plane, 402
  magnitude of, 155
  plane, 155
  position, 398
  scalar product of, 157
  scalar projection of, 157
  space, 398
  triple scalar product of, 401
  triple vector product of, 402
  unit, 156
  unit tangent, 159
  vector product of, 400
  vector projection of, 157
  velocity, 165
Vector functions:
  curl of, 427
  differentiation of, 158, 423
  divergence of, 427
  integration of, 427
Velocity:
  angular, 112
  in curvilinear motion, 165
  in rectilinear motion, 112, 247
Vertex of a parabola, 43
Volume:
  given by an iterated integral, 448
  of solids of revolution, 272
  under a surface, 448
  with area of cross section given, 280

Washer method, 273
Work, 301